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The Economics of Labor Coercion

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Abstract

The majority of labor transactions throughout much of history and a significant fraction of such transactions in many developing countries today are "coercive," in the sense that force or the threat of force plays a central role in convincing workers to accept employment or its terms. We propose a tractable principal-agent model of coercion, based on the idea that coercive activities by employers, or "guns," affect the participation constraint of workers. We show that coercion and effort are complements, so that coercion increases effort. Nevertheless, coercion is always "inefficient," in the sense of reducing utilitarian social welfare. Better outside options for workers reduce coercion, because of the complementarity between coercion and effort: workers with better outside option exert lower effort in equilibrium and thus are coerced less. Greater demand for labor increases coercion because it increases equilibrium effort. We investigate the interaction between outside options, market prices, and other economic variables by embedding the (coercive) principal-agent relationship in a general equilibrium setup, and study when and how labor scarcity encourages coercion. We show that general (market) equilibrium interactions working through prices lead to a positive relationship between labor scarcity and coercion along the lines of ideas suggested by Domar, while those working through outside options lead to a negative relationship similar to ideas advanced in neo-Malthusian historical analyses of the decline of feudalism. A third effect, which is present when investment in guns must be made before the realization of contracting opportunities, also leads to a negative relationship between labor scarcity and coercion. Our model also predicts that the slave trade makes slaves worse off, conditional on enslavement, and that coercion is more viable in industries that do not require relationship-specific investment by workers.

Keywords: coercion, feudalism, labor scarcity, principal-agent, slavery, supermodularity.
JEL Classification: D23, D74, D86, J01, P16.

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"In the context of universal history, free labor, wage labor, is the peculiar institution"—M. I. Finley[1]

1 Introduction

The standard economic models of the labor market, regardless of whether they incorporate imperfections, assume that transactions in the labor market are "free". For most of human history, however, the bulk of labor transactions have been "coercive," meaning that the threat of force was essential in convincing the worker to take part in the employment relationship, and thus in the determination of his or her compensation. Slavery and forced labor were the most common form of labor transactions in most ancient civilizations, including Ancient Greece, Ancient Egypt, Rome, several Islamic and Asian Empires, and most known pre-Colombian civilizations (e.g., Meltzer, 1993, Patterson, 1982, Lovejoy, 2000, Davis, 2006). Slavery was also the basis of the plantation economies in the Caribbean (e.g., Curtin, 1990, Klein and Vinson, 2007), in parts of Brazil and Colombia and in Southern United States (e.g., Patterson, 1982, Fogel and Engerman, 1974, Wright 1978), while forced labor played a major role in Spanish Latin America both in mining and in the encomiendas early on, and then in the subsequent hacienda system that developed in much of Latin America (e.g., Lockhart and Schwartz, 1983, Lockhart, 2000). While formal slavery has been rare in Europe since the middle ages, until the 19th century feudal labor relations, which include both forced labor services from serfs and various special dues and taxes to landowners, were the most important type of employment relationship except in the cities (e.g., Blum, 1998). Even today, the United Nations' International Labor Organization (ILO) estimates that there are over 12.3 million forced laborers worldwide (Andrees and Belser, 2009).[2]

The prevalence of slavery and forced labor in human history raises the question of when we should expect labor to be transacted in free markets rather than being largely or partly coerced. In a seminal paper, Domar (1970) provided one answer: slavery or serfdom should be more likely when labor is scarce so that (shadow) wages are high. This answer is both intuitive and potentially in line with the experience in the Caribbean, where Europeans introduced slavery into islands that had their population decimated during the early phases of colonization. In contrast, Acemoglu, Johnson and Robinson (2002) suggest and provided evidence for the view that Europeans were more likely to set up labor-coercive and extractive institutions when population density was high and labor was relatively abundant. Also consistent with this view, though working through a different mechanism and in a different context, is the relationship between population and feudalism emphasized by the famous "neo-Malthusian" theory of feudal decline, exemplified by Habakkuk (1958), Postan (1973, 2002), Leroy Ladurie (1974), and, from a somewhat different perspective, North and Thomas (1971). This theory claims that the coercive feudal labor relations started their decline when labor became

[1] From Finley (1976), quoted in Engerman (1999), p. 275.

[2] The ILO estimates that of these 12.3 million, 20% are coerced by the state (largely into military service), 14% are forced sex workers, and the remaining 66% are coerced by private agents in other industries, such as agriculture, mining, ranching, and domestic service (Andrees and Belser, 2009). Our model applies most directly to the last category.

scarce following the black death and other demographic shocks that reduced population and raised per capita agricultural income throughout Europe in the 16th century. The relationship between labor scarcity/abundance and coercion is also important in understanding the causes of continued prevalence of forced labor in many developing countries. There is also disagreement about the "efficiency" of slavery and other forced labor arrangements. Fogel and Engerman (1974) famously claimed that slavery in the US South was not unprofitable and that in fact it may have even been "efficient". This view contrasts strongly with those of Genovese (1976) and Williams (1944).

In this paper, we develop a simple model of coercive labor. In partial equilibrium, our model is a version of the principal-agent framework, with two crucial properties. First, the agent (worker) has no wealth so that there is a limited liability constraint, and the principal can punish as well as reward the agent. Second, the principal chooses the amount of "guns" (coercion), which influences the reservation utility (outside option) of the agent.[3] The first of these changes has been explored in several papers (e.g., Chwe, 1990, Dow, 1993, Sherstyuk, 2000). The second is, to our knowledge, new, and is crucial for our perspective on coercion and for all of our results; it captures the central notion that coercion is mainly about forcing workers to accept employment, or terms of employment, that they would otherwise reject.[4]

Our basic principle-agent model leads to several new insights about coercive labor relations. First, we show that coercion always increases the effort of the agent, which is consistent with Fogel and Engerman's view that Southern slavery was productive. In fact, we show that coercion may increase the level of effort above the first best (i.e., that which would have been observed in the absence of any information asymmetries). Second, we show that coercion is always "socially inefficient," because it involves an (endogenously) costly way of transferring resources (utility) from workers to employers. This result contrasts strongly with Fogel and Engerman's view that Southern slavery was "efficient".[5] Third, perhaps somewhat surprisingly, we find that workers with lower (ex ante) outside option are coerced more and provide higher levels of effort in equilibrium. The intuition for this result illustrates

[3] Throughout the paper, "guns" stand in for a variety of coercive tools that employers can use. These include the acquisition and use of actual guns by the employers as a threat against the workers or their families; the use of guards and enforcers to prevent workers from escaping or to force them to agree to employment terms favorable to employers; the confiscation of workers' identification documents; the setting up of a system of justice favorable to employers; investment in political ties to help them in conflictual labor relations; and the use of paramilitaries, strike-breakers and other non-state armed groups to increase their bargaining power in labor conflicts. In all of the instances of coercion mentioned here, for example, in the Caribbean plantation complex, African slave trade, the mita, the encomienda, the feudal system, and contemporary coercion in Latin America and South Asia, employers used several of these methods simultaneously.

[4] This view of coercion is consistent with the historical and contemporary examples given above as well as with the 1930 ILO Convention's definition of compulsory labor as "work or service which is exacted from any person under the menace of any penalty and for which the said person has not offered himself voluntarity," (quoted in Andrees and Belser, 2009, p. 179). Discussing contemporary coercion, Andrees and Belser (2009) write that "a situation can qualify as forced labor when people are subjected to psychological or physical coercion... in order to perform some work or service that they would otherwise not have freely chosen," (p. 179).

[5] It also offers a straightforward explanation for why gang labor disappeared after the Reconstruction, which was a puzzle for Fogel and Engerman. In this light, our model is much more consistent with Ransom and Sutch's evidence and interpretation of slavery and its aftermath in the South, in particular with their view that agricultural output collapsed in the post-bellum South due to the withdrawal of black labor (Ransom and Sutch, 1975, 1977, 1979). See also Irwin (1994) for an empirical investigation.

a central economic mechanism: in our model—and, we believe, most often in practice—effort and coercion are "complements". When the employer wishes to induce effort, he finds it optimal to pay wages after observing high output, so he must pay wages frequently when he induces high effort. Greater ex ante coercion enables him to avoid making these payments, which is more valuable when he must pay frequently, hence the complementarity between effort and coercion. This observation also implies that more "productive" employers will use more coercion, and therefore that a worker is worse off when matched with a more productive firm. This contrasts with the standard results in models of noncoercive labor markets, where ex post rent sharing typically makes workers matched with more productive employers better off. It also implies that coerced workers may receive high expected monetary compensation, despite having low welfare, which is consistent with the findings of both Fogel and Engerman and the ILO that coerced laborers often receive income close to that of comparable free laborers.[6]

The above-mentioned partial equilibrium results do not directly answer whether labor scarcity makes coercion more likely. To investigate this issue, and the robustness of our partial equilibrium results to general equilibrium interactions, we embed our basic principal-agent model of coercion in a general (market) equilibrium setting, with two distinct equilibrium interactions. The first is a *labor demand effect*: the price an employer faces for his output is determined endogenously by the production—and thus coercion and effort—decisions of all employers and affects the marginal product of labor, and the willingness of the employer to use coercion. The second is an *outside option effect*: workers' outside options affect the terms of their contracts and the level of coercion, and because agents who walk away from a coercive relationship (by paying the cost of doing so determined by the extent of coercion) may match with another coercive employer, effective outside options are determined by the overall level of coercion in the economy. Because of the complementarity of effort and coercion, these effects compete with each other, which makes the characterization of general equilibria a nontrivial task; formally, the set of equilibria may not be lattice (e.g., Topkis, 1998), so it is impossible to apply the standard methods used for comparative statics (e.g., Milgrom and Roberts, 1994, or Acemoglu and Jensen, 2009). Nevertheless, we show that a certain type of extremal equilibria has well behaved comparative statics, and we characterize these and show that they corroborate both the results of our principal-agent model and the results obtained in the case where only one of the general equilibrium channels is present.

More importantly, the general equilibrium model highlights various different channels linking labor scarcity to coercion. First, the labor demand effect makes coercion more likely when labor is scarce; this is reminiscent of Domar's intuition, whereby greater labor scarcity increases prices, and via this channel, raises the value of effort and thus encourages coercion.[7] Second, the outside option effect tends

[6] For example, the ILO constructed their 2009 estimate of the "Cost of Coercion" by estimating that forced workers are paid 80% of the prevailing minimum wage (ILO, 2009).

[7] A recent paper by Naidu and Yuchtman (2009) exploits the effects of labor demand shocks under the Master and Servant Acts in 19th-century Britain and finds evidence consistent with this labor demand effect.

to reduce coercion when labor is scarce. In particular, we endogenize outside options as a function of labor scarcity and show that greater outside options lead to lower equilibrium effort and lower coercion. This is similar in spirit to the neo-Malthusian theory of feudal decline, where labor scarcity increased the outside opportunities for serfs, particularly in cities, and led to the demise of feudal labor relations. Interestingly and importantly, our theory, though consistent with the neo-Malthusian view, is not subject to Brenner's (1976) famous critique of this approach based on the observation that falling populations did not lead to a decline in forced labor in every country and instance during this era, and in particular that in Eastern Europe they were associated with an increase in the prevalence of forced labor in the so-called "second serfdom" (see Ashton and Philpin, 1985). Our model in fact suggests that, since the "second serfdom" appears to have been driven by the increase in demand for agricultural goods from Western Europe, coercion should have increased during this episode. Moreover, the main mechanism that could have acted to reduce coercion during this period, the increased outside option of Eastern European workers, was probably relatively muted, because, in contrast to Western Europe, cities were less important as alternative employment opportunities for East European serfs.

Third, we also identify another channel via which labor scarcity reduces coercion, which we call the *economies of scale effect.* The main idea in Acemoglu, Johnson and Robinson (2002) is that setting up coercive and extractive institutions becomes more profitable when there are more numerous laborers to exploit. This can be captured in our model when employers choose their investment in guns ("coerciveness") before matching: when labor is abundant, investing in coercion is more likely to be useful, so coercion increases.

Finally, the general equilibrium analysis also enables us to study the welfare implications of "slave trade," whereby employers can sell their coerced agents to other potential employers. We show that such slave trade always reduces agent welfare and may reduce social welfare. This is because slave trade shifts the productivity distribution of active employers in the sense of first-order stochastic dominance, and with greater productivity comes greater coercion and lower worker utility.

The tractability of our principal-agent framework also enables us to investigate several extensions. First, we introduce ex ante investments and show that there is a type of "holdup" in this framework, where agents underinvest in skills that increase their productivity in their current coercive relationship (since their utility is decreasing in their productivity with their current employer) and overinvest in skills that increase their outside option (since this makes inducing effort more costly for the principal, which leads to lower coercion). This extension provides a potential (and new) explanation for why coercion is particularly prevalent in effort-intensive, low-skill labor, and relatively rare in activities that require investment in relationship-specific skills or are "care-intensive" (as argued by Fenoaltea, 1984). Second, we generalize the model so that coercion is exercised collectively by the producers (or by groups or law enforcement agencies under their control) and show that our main characterization and comparative static results extend to this case. Third, we investigate the consequences of a different type of holdup that emerges from the presence of coercion: the inability of the employer to commit

to payments, especially when he has many guns. Fourth, we show that coercion plays a more useful role when it affects the interim outside option of the agent, because it allows the principal to use ex post punishments, which may be useful because of the presence of the limited liability constraints. However, we also show that, under fairly weak assumptions, this effect cannot overturn our result that coercion is socially inefficient. Finally, we consider the case where effort and productivity are substitutes in a producer's production function and show that in this case, slave trade, conditional on the presence of coercion, may be beneficial by both increasing the productivity of active employers and reducing coercion, thus increasing slave welfare.

Despite the historical importance of coercion, the literature on coercive labor markets is limited. Early work in this area includes Conrad and Meyer (1958), Domar (1970), Fogel and Engerman (1974), and Ransom and Sutch (1977). Bergstrom (1971) defines a "slavery equilibrium," a modification of competitive equilibrium in which some individuals control the trading decisions of others, and shows existence and efficiency property of slavery equilibria. Findlay (1975) presents a model in which slave labor is produced mechanically by wages and policing of slaves, and analyzes how long it takes for a slave to purchase her freedom. Canarella and Tomaske (1975) also view a slave as a machine that produces output as a function of payments and force. They suggest that the profits of slaveholders is a poor proxy for slave welfare. Barzel (1977) performs a comparative analysis of slavery and free labor, under the assumption that slaves, but not free workers, must be monitored, and that slaves (exogenously) work harder. Fenoaltea (1984) posits that coercion can motivate "effort-intensive" but not "care-intensive" work and provides a historical discussion of slavery consistent with this view.[8]

In more recent work, Basu (1986) and Naqvi and Wemhöner (1995) develop models in which landlords may "coerce" their tenants by inducing other agents to ostracize them if they do not agree to favorable contract terms with the landlord. Genicot (2002) develops a model in which workers are collectively better off without the option of voluntarily entering into bonded labor agreements, as this stimulates the development of alternative forms of credit. Conning (2004) formalizes and extends Domar's hypothesis in the context of a neoclassical trade model with a reduced-form model of slavery, and provides extensive historical analysis. Lagerlöf (2009) analyzes a dynamic model of agricultural development, arguing that slavery is more likely at intermediate stages of development. In his model, as in Domar's paper, labor scarcity makes slavery relatively more likely than free labor.

The paper most closely related to ours is Chwe's (1990) important work on slavery. Chwe analyzes a principal-agent model closely related to our partial equilibrium model. There are several differences between Chwe's and our approaches. First, his model has no general equilibrium aspects and does not investigate the relationship between labor scarcity and coercion, which is one of our central objectives. Second, and more importantly, in Chwe's model, the principal cannot affect the agent's outside option, whereas all of our main results follow from our fundamental modeling assumption that coercion is

[8] See also Engerman and Sokoloff (1997) for a historical analysis based on the view that certain tasks are inherently better suited to slave labor. Andrees and Belser (2009) also argue that coercion is more common in labor-intensive sectors because these sectors are where saving on labor costs is most important.

about affecting the outside option of the agent (i.e., coercing an individual to accept an employment contract that he or she would not have otherwise accepted).[9] For example, this modeling assumption is important for our results on efficiency (in Chwe's model, coercion is typically efficiency-enhancing). Third, Chwe also imposes strong functional form assumptions to ensure the concavity of the principal's problem, whereas we establish and then exploit the supermodularity of the principal's problem.[10] This modeling strategy is central for our general equilibrium analysis.

The rest of the paper is organized as follows. Section 2 introduces our principal-agent model and the main features of the general equilibrium environments. Section 3 characterizes the solution to the principal-agent problem with coercion and presents several key comparative static results. Section 4 studies the general equilibrium model and investigates the relationship between labor scarcity and coercion. Section 5 presents several extensions. Section 6 concludes. Appendix A contains the proofs of Propositions 1 and 2 and Lemma 3 presented in Section 3. Appendix B briefly considers a generalization of the basic principal-agent model introduced in the next section to include multiple levels of output. Appendix C establishes the existence of equilibrium under the most general conditions considered in Section 4.

2 Model

In this section, we describe the environment. We start with the contracting problem between a "coercive" producer and an agent, and then describe how market prices and outside options are determined. We then provide a brief roadmap of the results that will be presented in the next two sections.

2.1 The Environment

There is a population of mass 1 of (coercive) *producers* and a population of mass $L < 1$ of *agents* (forced laborers, slaves, or simply workers); L is an inverse measure of labor scarcity in the economy. All agents are risk neutral. Each producer has a project that yields x units of a consumption good if successful, where x has a distribution $F(x)$ with continuous density $f(x)$ and support $[\underline{x}, \bar{x}]$, with $\underline{x} > 0$.[11] We refer to a producer with productivity x alternately as a producer of type x. All projects yields zero output if unsuccessful. All workers are identical.

Each producer is initially randomly matched with a worker with probability L. We first describe the actions and timing of events after such a match has taken place; we return to the matching stage

[9] Interestingly, Chwe agrees with this perspective and writes "one forces another person to be her labourer if one does so by (even indirectly) changing her reservation utility," (p.1110), but then assumes that the principal cannot change the agent's reservation utility. In particular, the agent in his model has exogenous reservation utility $\bar{U}$, and she receives at least this payoff in expectation under any contract she accepts.

[10] We do assume concavity of the principal's problem for one of our general equilibrium results (Proposition 5), imposing identical conditions to Chwe's to ensure concavity.

[11] The existence of a continuous density is assumed to simplify notation in the text and is relaxed in Appendix C. None of the results presented here are affected if F has mass points or is Dirac at some $\hat{x}$.

at the end of this section.

A producer with productivity x who is matched with an agent first chooses a level of guns, $g \geq 0$, at private cost $\eta\chi(g)$, and simultaneously offers a "contract" specifying an output-dependent wage-punishment pair (w^y, p^y) for $y \in \{y_l, y_h\}$, where $y_l = 0$, and, for a producer with productivity x, $y_h = x$.[12] We denote the punishment-wage pairs following high and low output by (w^h, p^h) and (w^l, p^l), respectively. We assume that the producer is fully committed to this wage-punishment pair (we return to this assumption in subsections 5.3 and 5.4). Wages and punishments have to be nonnegative, i.e., $w^y \geq 0$ and $p^y \geq 0$, so the agent has "limited liability". We also assume that $\chi(0) = 0$ and $\chi(\cdot)$ is twice differentiable, strictly increasing, and strictly convex, with derivative denoted by χ' that also satisfies $\chi'(0) = 0$ and $\lim_{g\to\infty} \chi'(g) = \infty$. The parameter $\eta > 0$ corresponds to the cost of guns, or to the cost of using coercion more generally, which is mainly determined by institutions, regulations and technology (e.g., whether slavery is legal). Following the contract offer of the producer, the agent either accepts or rejects the contract. If she rejects, then she receives payoff equal to her (intrinsic) outside option, $\bar{u}$, minus the level of guns, g, i.e.,

$$\bar{u} - g,$$

and the producer receives payoff 0. The interpretation of the agent's payoff is that, if she rejects the contract, the principal inflicts punishment g on her before she "escapes" and receives her outside option, $\bar{u}$.[13] This formulation introduces our main assumption that "coercion" (or "forced labor") involves using force or the threat of force to convince an agent to accept an employment relationship that she might have rejected otherwise. In light of this, throughout g will be our measure of *coercion*. We can also think of $\bar{u} - g$ as the agent's "extrinsic" outside option, influenced by the coercion of the producer, as opposed to $\bar{u}$, which could be thought of as the "intrinsic" outside option, determined outside the coercive relationship (e.g., in general equilibrium).

If the agent accepts the contract offer, then she chooses effort level $a \in [0, 1]$ at private cost $c(a)$. Here a is the probability with which the project succeeds, leading to output x. We assume that $c(0) = 0$ and that $c(\cdot)$ is strictly increasing, convex, and twice differentiable, with derivative denoted by c', and we also impose that $\lim_{a\to 1} c'(a) = \infty$ to ensure interior solutions. Suppose that the market price for the output of the producer is P. Thus when the agent accepts contract (w^y, p^y) and chooses effort a, guns equal g, and output equals y, the producer's payoff is

$$Py - w^y - \eta\chi(g),$$

and the agent's payoff is

$$w^y - p^y - c(a).$$

[12] In subsection 4.5, we consider the alternative timing in which producers choose g before the matching stage.

[13] This formulation implies that g affects the the agent's utility she "escapes" rather than her utility when she accepts the employment contract. Thus it is the threat of force, not its actual exercise, that matters. Nevertheless, since g is chosen at the beginning, this threat is only credible (feasible) when the producer undertakes the investments in coercive capacity ("guns").

These payoff functions also imply that punishment is costless for the producer, which is a simplifying assumption. An *equilibrium contract* (for given market price, P, outside option, $\bar{u}$, and producer type, x) is the subgame perfect equilibrium of the above-described game between the producer and the agent.[14] Given the timing of events, this equilibrium contract is given by the solution to the following maximization problem (for a producer with productivity x):

$$\max_{\left(a,g,w^h,w^l,p^h,p^l\right)\in[0,1]\times\mathbb{R}_+^5} a\left(Px-w^h\right)+(1-a)\left(-w^l\right)-\eta\chi(g) \tag{1}$$

subject to

$$a\left(w^h-p^h\right)+(1-a)\left(w^l-p^l\right)-c(a)\geq\bar{u}-g, \tag{IR$_0$}$$

and

$$a\in\arg\max_{\tilde{a}\in[0,1]}\tilde{a}\left(w^h-p^h\right)+(1-\tilde{a})\left(w^l-p^l\right)-c(\tilde{a}). \tag{IC$_0$}$$

Here (IR$_0$) can be interpreted as the "individual rationality" or "participation constraint" of the agent. If this constraint is not satisfied, then the agent would reject the contract—run away from the match with the producer. (IC$_0$) is the "incentive compatibility" constraint, ensuring that a is the agent's best response in the subgame following the contract offer and her acceptance of the contract. There is no loss of generality in letting the producer choose a from the set of maximizers, since if the producer expected an effort level choice from this set that did not maximize his payoff, then he would have a profitable deviation. Thus only solutions to this program are consistent with subgame perfect equilibria and all subgame perfect equilibria generate outcomes that constitute a solution to this program.

2.2 Market Interactions and General Equilibrium

To complete the description of the model, we next describe how the market price, P, and an agent's outside option, $\bar{u}$, are determined.

As noted above, matching between agents and producers is random, so the distribution of productivity among matched producers is identical to the unconditional distribution. Suppose that all producers with productivity x who are matched with agents choose coercion $g(x)$ and induce their agents to exert effort $a(x)$.[15] Then, average productivity among matched producers is

$$Q\equiv\int_{\underline{x}}^{\bar{x}}a(x)\,xdF(x). \tag{2}$$

[14] As our definition of general equilibrium below, Definition 1, makes it clear, not every equilibrium contract may be part of a (general) equilibrium, since the levels of P and $\bar{u}$, which we take as given here, may not correspond to general equilibrium values. One might thus alternatively refer to an equilibrium contract as an "optimal contract". We use the term equilibrium contract throughout for consistency.

[15] This implies that we are focusing on pure strategies. This focus is adopted in the text to simplify notation. Appendix C allows for mixed strategies. We also allow such mixed strategies in subsection 4.2, where all producers are assumed to have the same productivity, since in that context this assumption would be much more consequential. Also, note that $a(x)=0$ and $g(x)=0$ are allowed. Below we impose Assumption 1, which ensures that the effort level is always positive for all x.

The *aggregate level of production* is thus QL, and we assume that *market price* is given by

$$P = P(QL), \tag{3}$$

where $P(\cdot)$ is a decreasing demand schedule. Equation (3) captures the idea that greater output will reduce price. Equation (2) makes the equilibrium price P a function of the distribution of efforts induced by producers.

An agent's outside option, $\bar{u}$, is determined according to a reduced-form matching model. When an agent escapes, she either matches with another coercive producer or escapes matching with a coercive producer altogether (e.g., running away to the city or to freedom in the noncoercive sector). We assume that the probability that an agent who exercises her outside option matches with a randomly drawn, previously unmatched, coercive producer is $\gamma \in [0, 1)$, and the probability that she matches with an outside, noncoercive producer is $1 - \gamma$. In this latter case, she obtains an "outside option" $\tilde{u}$; we interpret $\tilde{u}$ as the wage in the noncoercive sector (and it will be endogenized in subsection 4.4 as a function of employment in the noncoercive sector). In practice, both the parameter γ and the exogenous outside option $\tilde{u}$ measure the possibilities outside the coercive sector. For example, both in the context of feudalism and in the context of forced agricultural labor relations, the existence of vibrant cities to which coerced workers may escape would correspond to a low γ and a high $\tilde{u}$.

This formulation implies that the outside option of an agent matched with a coercive producer, $\bar{u}$, satisfies

$$\bar{u} = \gamma \left(\int_{\underline{x}}^{\bar{x}} (\bar{u} - g(x))\, dF(x) \right) + (1 - \gamma)\, \tilde{u}. \tag{4}$$

Let G be the average number of guns used by (matched) coercive producers. Then, once again using the fact that the distribution of types of matched producers is the same as the unconditional distribution (because of random matching), the average number of guns is

$$G \equiv \int_{\underline{x}}^{\bar{x}} g(x)\, dF(x). \tag{5}$$

We refer to G as the *aggregate level of coercion* in the economy. (4) can now be written as

$$\bar{u} = \tilde{u} - \frac{\gamma}{1 - \gamma} G. \tag{6}$$

Intuitively, (6) states that an agent's outside option in the coercive sector equals her payoff from exiting the coercive sector minus the aggregate level of coercion, G, as given by (5), times a constant, $\gamma/(1 - \gamma)$, that is increasing in the difficulty of exiting the coercive sector.

Given this description, we now define a pure-strategy (general or market) equilibrium for this economy, referred to as *equilibrium* for short.[16] As noted above, we focus on pure-strategy equilibria in the text (except in subsection 4.2) to simplify notation, and relegate the discussion of mixed strategies to Appendix C. In particular, pure strategies allow us to focus on the functions $a : [\underline{x}, \bar{x}] \to \mathbb{R}_+$ and

[16] The term "market equilibrium" may be more appropriate since we do not endogenize the function $P(\cdot)$.

$g : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}_+$, which determine the level of effort induced and the level of coercion chosen by a producer of type x, to represent equilibrium behavior.[17]

Definition 1 *A (pure-strategy)* equilibrium *is a pair of functions* $(a^*(\cdot), g^*(\cdot))$ *such that, for each* $x \in [\underline{x}, \bar{x}]$, $(a^*(x), g^*(x))$ *is an equilibrium contract given market price* P *and outside option* $\bar{u}$, *and* P *and* $\bar{u}$ *are given by (3) and (6) evaluated at* $(a^*(\cdot), g^*(\cdot))$.

Throughout, we impose the following joint restriction on $P(\cdot)$, L, $\underline{x}$, $\tilde{u}$, and $c(\cdot)$:

Assumption 1

$$P\left(L\int_{\underline{x}}^{\bar{x}} xdF(x)\right)\underline{x} > \tilde{u} + c'(0).$$

Assumption 1 states that, even if all producers were to set $a(x) = 1$ and $g(x) = 0$, the marginal product of effort of an agent matched with the lowest-productivity producer would be greater than the agent's outside option plus her cost of effort at $a = 0$. Our analysis below will show that Assumption 1 is a sufficient (though not necessary) condition for all matched producers to induce their agents to exert positive effort (i.e., generate positive expected output) in equilibrium. Therefore, imposing this assumption allows us to focus on the economically interesting case and simplify the exposition considerably.

2.3 Roadmap

In the next section, we take the market price, P, and outside option, $\bar{u}$, as given and characterize equilibrium contracts for a producer of type $x \in [\underline{x}, \bar{x}]$. We then turn to the characterization of (general) equilibrium in Section 4, which will enable us to discuss issues related to the effects of labor scarcity on coercion, as well as to verify the robustness of the partial equilibrium effects in the presence of general equilibrium interactions.

Several important points, which are central for our equilibrium results in Section 4, will be already apparent from our "partial equilibrium" analysis. In particular, our model distinguishes between two possible channels through which labor abundance/scarcity, L, will affect the equilibrium. The first is the value of marginal product of effort, given by $P(QL)x$ for a producer has type x, and the second is the agent's outside option, $\tilde{u} - \gamma G/(1-\gamma)$. The overall effect of labor scarcity will work through both of these channels. For example, an increase in total labor (i.e., of agents that can be used in the coercive sector) will directly reduce the market price $P(QL)$, and we will see in the next section (Proposition 2) that this will reduce the amount of coercion in equilibrium contracts for any type x. This channel therefore captures Domar's basic insight that labor scarcity encourages coercion. In contrast, labor

[17] More generally, strategies would induce a probability distribution over a and g for each type. Allowing for such probability distributions is necessary for proving existence of equilibrium in the most general environment we consider below and also becomes necessary when the distribution of productivity, F, has mass points (see Appendix C). Focusing on pure strategies in the text saves on notation and simplifies the exposition.

scarcity may also affect $\bar{u}$, both via $\tilde{u}$ and via the aggregate level of coercion, G. For example, labor scarcity may increase $\tilde{u}$, and thus, from (6), $\bar{u}$. Since one of our main results is that higher $\bar{u}$ (better outside options) leads to lower coercion (see again Proposition 2), this general equilibrium mechanism implies that labor scarcity will be associated with lower coercion, capturing the ideas suggested by neo-Malthusian theories of the decline of feudalism, whereby greater outside wages make the coercion of serfs more difficult to maintain.

3 Equilibrium Contracts and Comparative Statics

3.1 Equilibrium Contracts

Recall that an equilibrium contract for a producer of type x is a solution to (1) subject to (IR_0) and (IC_0). Thus an equilibrium contract for a producer of type x is simply a tuple $\left(a, g, w^h, w^l, p^h, p^l\right) \in [0,1] \times \mathbb{R}^5_+$. Our first result provides a more tractable characterization of equilibrium contracts when they involve positive effort ($a > 0$). Throughout the paper, we use the notation $[z]_+ \equiv \max\{z, 0\}$.

Proposition 1 *Suppose $Px > \bar{u} + c'(0)$. Then any equilibrium contract involves $a > 0$ and $g > 0$, and an equilibrium contract for a producer of type x is given by $\left(a, g, w^h, w^l, p^h, p^l\right)$ such that*

$$(a, g) \in \arg \max_{(\tilde{a},\tilde{g})\in[0,1]\times\mathbb{R}_+} Px\tilde{a} - \tilde{a}\left[(1-\tilde{a})\,c'(\tilde{a}) + c(\tilde{a}) + \bar{u} - \tilde{g}\right]_+ - (1-\tilde{a})\left[-\tilde{a}c'(\tilde{a}) + c(\tilde{a}) + \bar{u} - \tilde{g}\right]_+ - \eta\chi(\tilde{g}), \tag{7}$$

with $w^l = p^h = 0$, $w^h = (1-a)\,c'(a) + c(a) + \bar{u} - g \geq 0$, and $p^l = ac'(a) - c(a) - \bar{u} + g \geq 0$.

Proof. See Appendix A. ■

Remark 1 The condition $Px > \bar{u} + c'(0)$ is automatically satisfied when Assumption 1 holds, since $P \geq P\left(L\int_{\underline{x}}^{\bar{x}} x dF(x)\right)$, $x \geq \underline{x}$ and $\bar{u} \leq \tilde{u}$. Thus Proposition 1 always holds under our maintained assumption. The qualifier $Px > \bar{u} + c'(0)$ is added for emphasis, since, as the proof illustrates, it ensures that $a > 0$, which is in turn important for this result. Problem (7) is not equivalent to the maximization of (1) subject to (IR_0) and (IC_0) when the solution to the latter problem involves $a = 0$.

Remark 2 Proposition 1 states that $(1-a)\,c'(a) + c(a) + \bar{u} - g \geq 0$ and $-ac'(a) + c(a) + \bar{u} - g \leq 0$ in any equilibrium contract; so in any equilibrium contract the right-hand side of (7) equals

$$Pxa - a(1-a)\,c'(a) - ac(a) - a\bar{u} + ag - \eta\chi(g)\,. \tag{8}$$

Remark 3 The maximization problem (7) is (weakly) supermodular in $(a, g, x, P, -\bar{u}, -\eta)$. We show in Appendix A (in particular, Lemma 7) that "generically" (or more precisely, for all parameter values, except for possibly one value of each parameter), the expression $-ac'(a) + c(a) + \bar{u} - g$ is *strictly* less than 0 in any equilibrium contract, and in this case, we show that (7) is *strictly* supermodular in $(a, g, x, P, -\bar{u}, -\eta)$ in the neighborhood of any equilibrium contract. This gives us strict rather than

weak comparative statics when we consider changes between any two possible parameter vectors (see Proposition 2 and Appendix A).

Proposition 1 simplifies our analysis considerably, since instead of solving a nonconvex constrained maximization problem, we can focus on an unconstrained supermodular maximization problem.

To obtain an intuition for Proposition 1, suppose that the solution to (1) (subject to (IR_0) and (IC_0)) indeed involves $a > 0$. Then recall that c is differentiable and that the first-order approach is valid in view of the fact that there are only two possible output realizations ($y \in \{y_l, y_h\}$), which implies that, given the contract offer (w^y, p^y), the agent's maximization problem in (IC_0) is concave. Moreover, recall that $\lim_{a \to 1} c'(a) = \infty$, so the solution involves $a < 1$. This implies that (IC_0) can be replaced by the corresponding first-order condition, where we write $u^h \equiv w^h - p^h$ and $u^l \equiv w^l - p^l$ for the agent's payoff (without effort costs) following the good and bad outcomes:

$$u^h - u^l = c'(a).$$

It wastes money for the producer to both punish and pay the agent after any outcome, so $p^h = 0$ if $u^h \geq 0$, $w^h = 0$ if $u^h \leq 0$, $p^l = 0$ if $u^l \geq 0$, and $w^l = 0$ if $u^l \leq 0$. Therefore, $w^h = [u^h]_+$ and $w^l = [u^l]_+$, so (1) can be reduced to the following constrained maximization problem:

$$\max_{(a,g,u^h,u^l) \in [0,1] \times \mathbb{R}_+^3} a\left(Px - \left[u^h\right]_+\right) - (1-a)\left[u^l\right]_+ - \eta\chi(g) \tag{9}$$

subject to

$$au^h - (1-a)u^l - c(a) \geq \bar{u} - g \tag{IR_1}$$

and

$$u^h - u^l = c'(a). \tag{IC_1}$$

Next, using (IC_1) to substitute for p^l in (IR_1) shows that this problem is equivalent to maximizing (9) subject to

$$u^h - (1-a)c'(a) - c(a) \geq \bar{u} - g. \tag{IR_2}$$

Finally, using (IR_2) to substitute u^h out of (9), and using (IR_2) and (IC_1) to substitute u^l out of (9), yields (7). Furthermore, it is intuitive that any solution to (7) will necessarily involve $w^l = 0$ and $w^h \geq 0$ if $a > 0$, so that the contract does not punish the agent for a good outcome and does not reward her for a bad outcome.

Note that (7) is not necessarily concave, so first-order conditions are not sufficient for characterizing an equilibrium contract. Nevertheless, it is concave in g for given a. This, combined with the assumption that $\chi'(0) = 0$ and $\lim_{g \to \infty} \chi'(g) = \infty$, implies that the first-order condition with respect to g, for a given equilibrium level of a,

$$\chi'(g) = \frac{a}{\eta}, \tag{10}$$

is necessary and sufficient whenever (7) is differentiable (i.e., whenever $u^l < 0$, which holds "generically," as shown in the proof of Proposition 2 and Lemma 7 in Appendix A). This immediately implies that a producer that wishes to induce higher effort will use more guns. Put differently, (7) is supermodular in a and g.[18] Though mathematically simple, this result is both important for our analysis and economically somewhat subtle. One might have presumed that high effort might be associated with less or more coercion. Our model implies that it will be associated with more coercion (but see also subsection 5.5). The logic is as follows: (IR_2) implies that coercion is valuable to the producer because it allows a one-for-one reduction in wages when the agent is successful (i.e., in w^h, since $u^h = w^h$ in an equilibrium contract), regardless of effort. An agent who exerts high effort succeeds more often and therefore must be rewarded more often. This makes coercion more valuable to the producer.

This observation will play an important role in the rest of the analysis, not only because it will lead to a positive covariation between a and g, but also because it implies that the set of equilibrium contracts will form a lattice (in the usual order on $\mathbb{R}^2$) and allows us to perform comparative static analysis on the set of equilibrium contracts in the weak set order,[19] even though (7) is not concave. In particular, let $(a^+(x), g^+(x))$ and $(a^-(x), g^-(x))$ be, respectively, the greatest and the least equilibrium contracts for a producer of type x (given P, $\bar{u}$, and η). These are well-defined because, given the supermodularity of (7) in (a, g), the set of solutions is a lattice. The analysis of the behavior of the set of equilibrium contracts in the weak set order is equivalent to the analysis of the behavior of the pair of contracts $(a^+(x), g^+(x))$ and $(a^-(x), g^-(x))$ (provided that they move in the same direction).[20]

Next, recall from Remark 3 that (7) is also supermodular in $(a, g, x, P, -\bar{u}, -\eta)$ (and thus it exhibits increasing differences in (a, g) and $(x, P, -\bar{u}, -\eta)$). This implies that changes in productivity, x, market price, P, outside option, $\bar{u}$, and cost of guns, η, will have unambiguous effects on the set of equilibrium contracts. This observation enables us to derive economically intuitive comparative static results from standard monotonicity theorems for supermodular optimization problems (e.g., Topkis, 1998). This discussion thus establishes the following important result:

Proposition 2 *The set of equilibrium contracts for a producer of type x forms a lattice, with greatest and smallest equilibrium contracts $(a^+(x), g^+(x))$ and $(a^-(x), g^-(x))$. The extremal equilibrium contracts $(a^+(x), g^+(x))$ and $(a^-(x), g^-(x))$ are increasing in x and P and decreasing in $\bar{u}$ and η.*

Proof. See Appendix A. ■

[18] Formally, (10) only shows that (7) is supermodular in a and g in the region where $u^l < 0$. However, direct inspection of (7) shows that it is (weakly) supermodular everywhere.

[19] Recall that $X \geq X'$ in the weak set order if for every $x \in X$ there exists $x' \in X'$ with $x \geq x'$, and for every $x' \in X'$ there exists $x \in X$ with $x \geq x'$.

[20] If X and X' are lattices, $X \geq X'$ in the weak set order if and only if the highest element of X is higher than the highest element of X', and the lowest element of X' is lower than the lowest element of X.

The results contained in Proposition 2 are intuitive. Higher x and P both increase the value of a successful outcome for the producer and thus the value of effort. Since effort and coercion are complements, both a and g increase. This is the essence of the labor demand effect, which was discussed in the Introduction and will be studied in greater detail in the next section. Higher costs of guns, η, makes coercion less attractive, and thus tends to reduce g. The complementarity of effort and guns then raises the marginal cost of inducing higher effort, since now the producer has to pay higher wages to satisfy (IR_2). Thus an increase in η reduces both a and g. The intuition for the effect of outside options is given in subsection 3.3.

3.2 Discussion of Assumptions

It is useful to briefly discuss the role of various assumptions in leading to the sharp characterization result in Proposition 1 and to (7), which will play a central role in the rest of our analysis. Eight assumptions deserve special mention. First, we assume that the coercive relationship starts with a match between the producer and the agent, and the only reason for the producer to offer an "attractive" contract to the agent is to prevent her from running away. This is important for our analysis, since it implies that producers do not compete with each other in order to attract agents. We believe that this is a realistic assumption in the context of coercion. Serfs in Europe and forced laborers in Latin America were often tied to the land and employers did not need to attract potential workers into serfdom. Slaves throughout the ages were often captured and coerced. According to Andrees and Belser (2009), even today many forced employment relationships originate when employers are able to lure workers into such relationships, for example by promising good working conditions that do not materialize once workers arrive at a plantation or mine, at which point they are not allowed to leave.

Second, we use a principal-agent model with moral hazard. Third, we impose a limited liability constraint, so that the worker cannot be paid a negative wage. We view both of these assumptions as central for a good approximation to actual coercive relationships. Inducing agents to exert effort is a crucial concern in coercive employment relationships, and clearly these agents cannot make (unlimited) payments to their employers, since they are trapped in the coercive relationship without other sources of income. From a theoretical point of view, both of these assumptions are important for our results (and we view this as a strength of our approach in clearly delineating results that depend on distinctive features of coercive relationships). Relaxing either of these two assumptions would imply that the employer could implement the "first-best" level of effort, $a_{FB}(x)$, given by $Px = c'(a_{FB}(x))$, either by dictating it or by choosing large enough negative payments after low output (given risk neutrality). In particular, in this case the problem of a coercive producer, with productivity x, could be written as $\max_{(g,w^h)} a_{FB}(x)\left(Px - w^h\right) - \eta\chi(g)$ subject to $a_{FB}(x)\, w^h - c(a_{FB}(x)) \geq \bar{u} - g$. Since the constraint will necessarily hold as equality, this problem can be written as

$$\max_{g\geq 0} a_{FB}(x)\, Px - (\bar{u} - g + c(a_{FB}(x))) - \eta\chi(g).$$

This problem is no longer strictly supermodular and coercion will always be independent of both $\bar{u}$

and P. Therefore, all of our results depend on the principal-agent approach and the importance of effort and moral hazard (and limited liability).

Fourth, we allow the principal to use punishment $p \geq 0$. The presence of such punishments is another realistic aspect of coercive relationships. Moreover, they play an important role in our theoretical results by ensuring that the participation constraint, (IR_0) or (IR_2), hold as equality. In the absence of such punishments, the participation constraint can be slack, in which case there would be no role for using our measure of coercion, g, to reduce the (extrinsic) outside option of the agent. One could construct different versions of the principal-agent problem, where the participation constraint holds as equality even without punishments, and we conjecture that these models would generate similar insights. Our formulation is particularly tractable and enables us to focus on the key role of coercion in inducing agents to accept employment terms that they would not have done in the absence of coercion or threats of coercion. We also view a situation in which the participation constraint of the agent is binding as a good approximation to coercive relationships; if the participation constraint were slack, we could not talk of a situation in which coercion is used for making agents accept employment terms that they would not otherwise accept. Instead, one might imagine a situation in which an interim participation constraint is slack; in subsection 5.4, we show how this can be incorporated into our framework and demonstrate that it does not change the general economic insights derived in the rest of the analysis.

Fifth, we impose Assumption 1 throughout. This assumption implies that productivity in the coercive sector is (sufficiently) greater than $\tilde{u}$ and thus greater than agents' (intrinsic) outside option, $\bar{u}$. This makes coercive relationships viable, and corresponds to situations in which coercive producers have access to valuable assets for production, such as land or capital. This type of unequal access assets of production is a key feature supporting coercive relationships such as serfdom, forced labor, or slavery.

Sixth, we assume that coercion is undertaken by each producer, and thus corresponds to the producer's use of armed guards, enforcers or threat of violence against its laborers. In practice, much coercion is undertaken jointly by a group of producers (for example, via the use of local or national law enforcement, or the judiciary system, as was the case in the US South both before and after the Civil War, e.g., Key, 1949, or Ransom and Sutch, 1977). Moreover, even coercion by each individual producer presumes an institutional structure that permits the exercise of such coercion. A comprehensive study of coercion requires an analysis of the politics of coercion, which would clarify the conditions under which producers can use the state or other enforcement mechanisms to exercise coercion and pass laws reducing the outside option of their employees. Our analysis is a crucial step towards this bigger picture, since it clarifies the incentives of each producer to use coercion before incorporating details of how they will solve the collective action problem among themselves and cooperate in coercive activities. To highlight that this first step captures and clarifies much of the economics of coercion, in subsection 5.2, we show that our major comparative static results

immediately generalize to an environment in which producers make voluntary contributions to a collective coercion technology.

Seventh, we assume bilateral risk-neutrality. The effects we focus on in this paper do not disappear in the presence of risk-aversion, though adding risk-aversion complicates the analysis, as one might expect. Nevertheless, there is at least one important way in which making the agent risk-averse reinforces our central intuition that effort and coercion are complementary. Consider the case where $u^l < 0$, so that the sole purpose of coercion is to reduce w^h. By (IR_2) and convexity of $c(\cdot)$, u^h is increasing in a (for fixed g), and increasing g allows the principal to reduce u^h one-for-one. When the agent is risk-averse, the wage that the producer must pay after high output to give the agent utility u^h is convex in u^h, since u^h is a concave function of w^h. Therefore, reducing u^h is more valuable to the principal when a is higher, which provides a second source of complementarity between a and g in the principal's problem. Therefore, our general presumption is that risk-aversion should strengthen our results.[21]

Finally, we assume only two levels of output. This is for simplicity, and Appendix B shows how our results can be generalized, under (relatively mild) additional assumptions, to an environment with multiple levels of output.[22]

3.3 Further Comparative Statics

Despite the large historical literature on slavery and the importance of the issue of coercion in labor markets both in the past and today, little is understood about the relationship between coercion and other economic variables. For example, does labor scarcity increase coercion? Does coercion increase effort, productivity and output? Does it increase or reduce "economic efficiency"? Do better outside options for workers lead to more coercion?[23] Although the first of these questions must await our general equilibrium analysis in the next section, Proposition 2 enables us to provide comprehensive answers to the remaining questions and others. Throughout this subsection, we simplify the exposition by simply stating that "a variable z increases coercion or effort" to mean that the greatest and least

[21] The reason that this argument is not completely general is that u^l may equal 0 in an equilibrium contract. However, if the agent's utility function for money, $u(w)$, satisfies $u'''(w) < 0$, it can be shown that the producer's problem, the analogue of (7), is strictly supermodular in (a, g) regardless of u^l (proof available from the authors upon request). In fact, the producer's problem remains strictly supermodular in this case even in the absence of limited liability and/or punishments. Thus, risk-aversion provides an alternative route to our key comparative static results, provided one is willing to make the somewhat unappealing assumption that $u'''(w) < 0$.

[22] More specifically, we show that provided that the first-order approach is valid (which, given the agent's risk-neutrality, is not very restrictive when we use Jewitt's (1988) sufficient conditions), the multiple-output analog of (7) is supermodular if the ratio of the likelihood ratio evaluated at the highest possible value of y to the likelihood ratio evaluated at the lowest possible value of y is increasing in a. Intuitively, this condition guarantees that the effect that reducing w^h is more valuable to the producer when a is high is not offset by the new possibility that an increase in a leads to a decrease in the amount by which a given increase in g allows the principal to reduce w^h. The details are provided in Appendix B.

[23] The only previous answer to this question in a contract-theoretic model is provided by Chwe (1990), who shows that, under assumptions that ensure concavity of the producer's problem, better outside options lead to higher payoffs after both output realizations for the agent. Because higher payoffs are associated with less ex post punishment, this can be interpreted as better outside options leading to less "coercion". In Chwe's model, this result depends on the agent's risk-aversion (i.e., on "income effects").

equilibrium contracts, $(a^{+}(x), g^{+}(x))$ and $(a^{-}(x), g^{-}(x))$, are increasing in z.

First, Proposition 2 immediately formalizes the outside option effect on coercion discussed in the Introduction (proof omitted):

Corollary 1 *An agent with a worse (intrinsic) outside option (i.e., lower $\bar{u}$) is subject to more coercion.*

This result is at first surprising. Since higher g offsets the effect of higher $\bar{u}$ (recall (IR_0) or (IR_2)), one might have expected g and $\bar{u}$ to covary positively. This presumption would also follow from a possible, perhaps mistaken, reading of Domar (1970), on the hypothesis that labor scarcity corresponds to higher $\bar{u}$. Corollary 1 shows that the opposite is always the case. The intuition for this result is interesting: An individual with a worse outside option (lower $\bar{u}$) will be induced to work harder, because her participation constraint, (IR_2), is easier to satisfy. As discussed in the previous subsection, agents working harder will be successful more often and will be paid more often. This increases the value of coercion to the producer. Thus a lower outside option, $\bar{u}$, is associated with more coercion in equilibrium contracts.

Next, we turn to the implications of coercion on worker effort and productivity. To conduct comparative statics with respect to coercion, we change the cost of coercion, η. Throughout the paper, when we make comparisons between a coercive equilibrium contract ("coercion") and "no coercion," the latter refers to a situation in which either we exogenously impose $g = 0$ or, equivalently, $\eta \to \infty$. Given this convention, the next corollary is also an immediate implication of Proposition 2 (proof omitted):

Corollary 2 *Coercion (or cheaper coercion, i.e., lower η) increases effort.*

This result may explain Fogel and Engerman's (1974) finding that productivity was high among slaves in the US South in the antebellum period. It is also intuitive. Coercion and effort are complements, and when the former becomes cheaper, equilibrium contracts induce greater effort. Conversely, if coercion is banned or made prohibitively expensive, the producer will have to pay higher wages to satisfy (IR_0) and will choose to induce lower levels of effort.

The next corollary is immediate from the analysis in subsection 3.1 and shows that coercion is unambiguously bad for the welfare of the agent.[24]

Corollary 3 *Coercion (or cheaper coercion, i.e., lower η) reduces agent welfare.*

Proof. From the analysis leading up to Proposition 1, (IR_0) holds as equality, and thus the welfare of the agent under any equilibrium contract is equal to $\bar{u} - g$. The result follows from the fact that g is decreasing in η. ∎

[24] This contrasts with Chwe's (1990) framework, where the agent receives her outside option (reservation utility) regardless of coercion.

Even though coercion reduces agent welfare, it may still increase some measures of "economic efficiency" or net output. Fogel and Engerman (1974), in fact, not only documented that slaves in the US South had relatively high productivity, but argued that the slavery system was "economically efficient," perhaps in the sense of increasing either output net of effort costs or social welfare. Note that these differ in our model since the former includes neither the cost of guns nor the welfare costs to the agent of punishments; that is, *net output* is defined as $Px - c(a)$, while (utilitarian) *social welfare* is defined as $Pxa - a(1-a)c'(a) - ac(a) - a\bar{u} + ag - \eta\chi(g) + \bar{u} - g$ (by (7), this is the sum of the producer's and worker's utilities). We now examine how these quantities are affected by coercion. One might conjecture that, since effort under moral hazard is typically below first-best, coercion, which increases effort (Corollary 2), should increase an appropriate notion of economic efficiency. This is not the case, however, as shown by the next two corollaries.

First, we show that coercion can lead to effort *above* the first-best level that would prevail in the absence of information asymmetries and limited liability constraints (i.e., a_{FB} given by $c'(a_{FB}) = Px$); this implies that coercion can in fact reduce net output.[25] The argument leading to this result is simple. Since $\lim_{a\to 1} c'(a) = \infty$, the first best effort, a_{FB}, is strictly less than 1. We show that as $\eta \to 0$, any equilibrium contract involves an effort level a arbitrarily close to 1 (see the proof of Corollary 4). The intuition for this is that coercion allows the producer to "steal utility from the agent," as shown by (IR_0) or (IR_2). Moreover, since the agent is subject to limited liability, the transfer of utility from the agent to the producer will take place "inefficiently," by inducing excessive effort. The next corollary formalizes this argument.

Corollary 4 *There exists $\eta^{**} > 0$ such that if $\eta < \eta^{**}$, then any equilibrium contract involves effort a strictly greater than a_{FB}.*

Proof. From the proof of Proposition 2 in Appendix A, there exists $\eta^* > 0$ such that $u^l < 0$ in any equilibrium contract with $a > 0$ if $\eta \leq \eta^*$. Therefore, if $\eta \leq \eta^*$, an equilibrium contract involves an effort level that solves

$$\max_{\tilde{a}\in[0,1]} Px\tilde{a} - \tilde{a}\left[(1-\tilde{a})c'(\tilde{a}) + c(\tilde{a}) + \bar{u} - (\chi')^{-1}\left(\frac{\tilde{a}}{\eta}\right)\right]_+ - \eta\chi\left((\chi')^{-1}\left(\frac{\tilde{a}}{\eta}\right)\right),$$

where we have used the fact that, when $u^l < 0$, $\chi'(g) = a/\eta$. From Proposition 1, every solution a is strictly positive, for all $\eta \geq 0$, and $(1-a)c'(a) + c(a) + \bar{u} - (\chi')^{-1}\left(\frac{a}{\eta}\right) > 0$. By Proposition 2, the smallest and greatest solutions are decreasing in η. For $\eta > 0$, let $\underline{a}(\eta) > 0$ be the smallest solution, and note that as η decreases, $\underline{a}(\eta)$ is nondecreasing, and thus remains uniformly bounded away from

[25] Technically, the fact that coercion can lead to effort above the first-best level does not immediately imply that coercion can reduce net output, as it is possible that effort without coercion is far below the first-best level. However, the proof of Corollary 4 shows that, as $\eta \to 0$, all equilibrium contracts under coercion have $a \to 1$, which implies that net output under coercion converges to $-\infty$. So the statement that coercion can reduce net output is certainly true, and in fact coercion *must* reduce net output if η is very low.

0. Next, since the maximand is right-differentiable and $\lim_{a\to 1} c'(a) = \infty$, the first-order condition

$$(1 - \underline{a}(\eta))\left(c'(\underline{a}(\eta)) + \underline{a}(\eta)\, c''(\underline{a}(\eta))\right) + c(\underline{a}(\eta)) + \bar{u} = (\chi')^{-1}\left(\frac{\underline{a}(\eta)}{\eta}\right) + Px$$

must hold. Now consider $\eta \to 0$. For any $\underline{a}(\eta) < 1$ the left-hand side is finite, while, since χ is convex and satisfies $\lim_{g\to\infty} \chi'(g) = \infty$, the right-hand side converges to ∞. This implies that $\underline{a}(\eta)$ must also converge to 1, and thus all solutions must converge to 1 as η converges to zero. Since $a_{FB} < 1$, this completes the proof of the corollary. ■

The second part of the answer to the question of whether coercion can increase "economic efficiency" involves looking at (utilitarian) social welfare. The next corollary shows that this measure of social welfare is *always lower* under coercion.

Corollary 5 *Social welfare in any equilibrium contract under coercion ($\eta < \infty$) is strictly lower than social welfare in any equilibrium contract under no coercion.*

Proof. Let (a, g) be the effort and coercion involved in an equilibrium contract under coercion. Let SW^C be social welfare under coercion given (a, g), and let SW^N be the infimum over all equilibrium contracts of social welfare under no coercion. We have

$$\begin{aligned} SW^C &= Pxa - a(1-a)\,c'(a) - ac(a) - a\bar{u} + ag - \eta\chi(g) + \bar{u} - g \\ &< Pxa - a(1-a)\,c'(a) - ac(a) - a\bar{u} + \bar{u} \\ &\leq \max_{\tilde{a}\in[0,1]} Px\tilde{a} - \tilde{a}(1-\tilde{a})\,c'(\tilde{a}) - \tilde{a}c(\tilde{a}) - \tilde{a}\bar{u} + \bar{u} \\ &= SW^N, \end{aligned}$$

where the second and third lines are immediate since $g > 0$ by (10) and $a \leq 1$, and the fourth line follows because the maximand in the third line is the same as the maximand in (7), i.e., as (8), with g set to zero, which, by definition, characterizes the set of equilibrium contracts under no coercion. ■

The intuition for Corollary 5 is simple: coercion is a costly means of transferring utility from the agent to the producer. Therefore, it is necessarily overused in equilibrium. Despite the simplicity of this intuition, the result contained in Corollary 5 has not appeared in the literature to the best of our knowledge, because the central role of coercion in affecting the participation constraint has been overlooked. This omission is also responsible for Fogel and Engerman's inference from the observation that slaves exert high effort that the slavery system may have been "economically efficient". In fact, the inefficiency of coercion in the labor market is stronger than that implied by Corollary 5. In the next section, we show that coercion can be Pareto inefficient in the sense that not only social welfare decreases, but the equilibrium allocation involves lower welfare both for producers and for agents relative to an allocation where $g = 0$ (or where $\eta \to \infty$).

Another immediate implication of Proposition 2 is the following (proof omitted):

Corollary 6 *A coerced worker is better off when matched with a less productive producer (i.e., a producer with lower x).*

The intuition (and the proof) is simply that producers with higher x use more coercion and thus give lower welfare, $\bar{u} - g$, to their agents. Once again, though straightforward, this corollary has interesting economic implications. One of these, which will be discussed further in the next section, is that the slave trade makes agents worse off, even conditional on their being coerced.

Finally, we consider the cross-sectional relationship between coercion and expected incentive pay, assuming that cross-sectional variation is generated by variation in x. Proposition 1 implies that an equilibrium contract always involves $w^l = 0$ and

$$w^h = \bar{u} - (\chi')^{-1}\left(\frac{a}{\eta}\right) + (1-a)\,c'(a) + c(a).$$

An increase in x leads to an increase in a, and affects w^h only through its effect on a. We have

$$\begin{aligned}\frac{\partial w^h}{\partial a} &= -\frac{1}{\eta\chi''\left((\chi')^{-1}\left(\frac{a}{\eta}\right)\right)} + (1-a)\,c''(a)\\ &= -\frac{1}{\eta\chi''(g)} + (1-a)\,c''(a).\end{aligned}$$

The sign of this derivative is ambiguous: the direct effect of an increase in a is to increase w^h (as (IR_2) binds), but an increase in a also increases g (through (10)), which reduces w^h (again through (IR_2)). If the first effect dominates, then an increase in x leads to higher extremal values of g and w^h; if the second effect dominates, then an increase in x leads to higher extremal values of g and lower extremal values of w^h. The former case is particularly interesting because it provides an explanation for Fogel and Engerman's observation that workers who are subjected to more coercion are not necessarily less well-paid; and yet, in contrast to their interpretation, our result also shows that this has nothing to do with the efficiency of slavery. We state this result in the next corollary (proof in the text):

Corollary 7 *Cross-sectional variation in x leads to a positive correlation between (the greatest and smallest equilibrium contract values of) g and w^h if $\partial w^h/\partial a > 0$ for all a, and leads to a negative correlation between (the greatest and smallest equilibrium contract values of) g and w^h if $\partial w^h/\partial a < 0$ for all a.*

4 General Equilibrium

In this section, we characterize and discuss the comparative statics of equilibria as defined in Definition 1. Our main objectives are twofold. The first is to understand the relationship between labor scarcity and coercion, which was one of our main motivating questions. The second is to investigate the robustness of the partial equilibrium insights derived in the previous section, including those concerning welfare.

We first recall that Assumption 1 ensures that

$$P(QL)\,x > \bar{u} + c'(0) = \tilde{u} - \frac{\gamma}{1-\gamma}G + c'(0),$$

for all $x \in [\underline{x}, \bar{x}]$. This implies that Proposition 1 applies and characterizes the set of equilibrium contracts given P and $\bar{u}$ (or, alternatively, G). The characterization of a (general) equilibrium then simply amounts to finding the set of contracts (mappings) $(a^*(\cdot), g^*(\cdot))$ satisfying (7), where P and $\bar{u}$ are given by (3) and (6) evaluated at $(a^*(\cdot), g^*(\cdot))$.

At first, this appears to be a relatively simple equilibrium problem, since both endogenous (general equilibrium) objects, P and $\bar{u}$, depend on the strategy profile of producers via two aggregates, Q and G, as defined in (2) and (5). Therefore, this is an aggregative game (e.g., Acemoglu and Jensen, 2009). However, in the most general case, the structure of equilibria in this game is complex, because the game has multiple equilibria and exhibits neither strategic complements nor strategic substitutes. Loosely speaking, when a set of producers choose higher (a, g), this increases both Q and G, but the increase in Q reduces P (since the function $P(\cdot)$ is decreasing) and discourages others from increasing their (a, g), while the increase in G reduces $\bar{u}$ and encourages further increases in (a, g). These interactions, in fact, imply that the set of equilibria may not be a lattice, making the characterization of equilibrium comparative statics a challenging task.

This challenge notwithstanding, we provide a fairly tight characterization of the structure of equilibria and their comparative statics, which generalize the results provided in Proposition 2. We also show how our framework flexibly accommodates the three different channels from labor scarcity to coercion discussed in the Introduction. We now provide a roadmap for the rest of the section. For the first three subsections, we focus on the model as introduced in Section 2. In the next subsection, we assume that only one of the two aggregates, Q and G, is relevant, in the sense that either $P(\cdot) \equiv \bar{P}$ or $\gamma = 0$. In this case, the characterization of the set of equilibria becomes tractable.[26] In subsection 4.2, we show that another special case, in which all producers have the same productivity x and (7) is concave, also leads to sharp comparative static results. In subsection 4.3, we develop a more general approach to the characterization of equilibria and provide comparative static results.[27] These results do not involve the analysis of extremal equilibrium (Q, G) pairs (as these may not exist), but focus on extremal equilibria either for Q or for G, and show that equilibria always change in a way that shifts these extremal equilibria unambiguously in the direction of the partial equilibrium results.

Beyond proving existence of equilibrium and generalizing the partial equilibrium results of the previous section, our treatment in subsections 4.1-4.3 also provides the first answer to the question

[26] In the first case, we end up with a game with strategic complements (as in Milgrom and Roberts, 1994) and standard theorems ensure both existence of equilibrium and comparative statics generalizing the partial equilibrium results from the previous section. In the latter case, we have an aggregative game with strategic substitutes, and now results from Acemoglu and Jensen (2009) can be applied to derive similar results. However, instead of appealing to these theorems, we develop a new, simple approach for characterizing extremal equilibria and comparative statics in both cases. This approach is useful because it will be used also in subsections 4.2 and 4.3, where neither of these existing methods can be used for analysis.

[27] Existence of equilibria in the general model of subsection 4.3 is established in Appendix C.

concerning the relationship between labor scarcity and coercion. Except in the case in which $P(\cdot) \equiv \bar{P}$, labor scarcity increases the market price for the output of the producers and thus creates a *labor demand effect*, increasing coercion (thus formalizing and extending Domar's (1970) intuition). In subsection 4.4, we modify our baseline model of Section 2 by allowing $\tilde{u}$ to depend on employment in the noncoercive sector. Labor scarcity now also reduces employment in the noncoercive sector and tends to increase $\tilde{u}$ via an (equilibrium) *outside option effect*. Now, supposing that $P(\cdot) \equiv \bar{P}$, comparative statics with respect labor scarcity are reversed, because a more limited supply of labor increases outside options and discourages coercion.[28] This result formalizes and extends the intuition of the neo-Malthusian theories of the decline of feudalism. In subsection 4.5, we consider another modification of our original model in which producers choose guns before matching. In this case, labor scarcity discourages coercion via an *economies of scale* effect, since a given investment in guns is less useful when there is less labor to coerce (i.e., a lower probability of matching with an agent that can be coerced). This mechanism is related to the ideas suggested in Acemoglu, Johnson and Robinson (2002) for why Europeans were more likely to introduce coercive institutions in colonies with greater population density. Finally, in subsection 4.6, we revisit the welfare analysis in general equilibrium and show that in addition to the result that coercion reduces (utilitarian) social welfare, it may in fact be "Pareto inefficient" (harming producers themselves as well as workers). We also show that slave trade, even conditional on the presence of coercion, reduces the welfare of workers.

4.1 Single Relevant Aggregate

In this subsection we derive comparative statics under the assumption that either $P(\cdot) \equiv \bar{P}$ (i.e., $P(QL) = \bar{P}$ for all Q) or $\gamma = 0$. This implies that only one of the two aggregates, Q and G, is relevant in problem (7). Let $(a^-, g^-)(x, Q, G)$ and $(a^+, g^+)(x, Q, G)$ denote the smallest and greatest solutions to (7), respectively, as functions of x, Q, and G.

Suppose that $P(\cdot) \equiv \bar{P}$, so that (7) can be rewritten as

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} a\bar{P}x - a\left[(1-a)\,c'(a) + c(a) + \tilde{u} - \frac{\gamma}{1-\gamma}G - g\right]_+ - (1-a)\left[-ac'(a) + c(a) + \tilde{u} - \frac{\gamma}{1-\gamma}G - g\right]_+ - \eta\chi(g).$$

This does not depend on Q, so we may rewrite $(a^-, g^-)(x, Q, G)$ and $(a^+, g^+)(x, Q, G)$ as $(a^-, g^-)(x, G)$ and $(a^+, g^+)(x, G)$. Note that Proposition 2 implies that both of these are increasing in both x and G. Now define

$$\phi\left(G, \bar{P}, \tilde{u}, \gamma, \eta\right) \equiv \left[\int_{\underline{x}}^{\bar{x}} g^-(x, G)\, dF(x),\, \int_{\underline{x}}^{\bar{x}} g^+(x, G)\, dF(x)\right].$$

That is, $\phi\left(G, \bar{P}, \tilde{u}, \gamma, \eta\right)$ is the interval with endpoints equal to the greatest and smallest aggregate levels of coercion consistent with producers' offering equilibrium contracts given aggregate level of

[28] Naturally, if both $\tilde{u}(\cdot)$ and $P(\cdot)$ are non-constant, then there will be two counteracting general equilibrium effects and the overall impact of labor scarcity will be ambiguous.

coercion G. We will write $\phi(G)$ for $\phi\left(G, \bar{P}, \tilde{u}, \gamma, \eta\right)$ when the parameters are understood. It is clear that if G is an equilibrium aggregate level of coercion, then G is a fixed point of $\phi(G)$. The converse is also true:

Lemma 1 *If $G \in \phi(G)$, then G is an equilibrium aggregate level of coercion.*

Proof. If $G \in \phi(G)$, then, by the Intermediate Value Theorem, there exists $x^* \in [\underline{x}, \bar{x}]$ such that

$$G = \int_{\underline{x}}^{x^*} g^-(x, G)\, dF(x) + \int_{x^*}^{\bar{x}} g^+(x, G)\, dF(x).$$

The strategy profile in which producers of type $x \leq x^*$ choose $(a^-, g^-)(x, G)$ and producers of type $x > x^*$ choose $(a^+, g^+)(x, G)$ is an equilibrium. ■

The next lemma shows that if G_0 is the smallest [greatest] fixed point of $\phi(G)$, then G_0 is the smallest [greatest] element of the set $\phi(G_0)$.

Lemma 2 *If G^- is the smallest fixed point of $\phi(G)$, then $G^- = \int_{\underline{x}}^{\bar{x}} g^-(x, G^-)\, dF(x)$. If G^+ is the greatest fixed point of $\phi(G)$, then $G^+ = \int_{\underline{x}}^{\bar{x}} g^+(x, G^+)\, dF(x)$.*

Proof. Suppose G^- is the smallest fixed point of $\phi(G)$. Then $G^- \geq \int_{\underline{x}}^{\bar{x}} g^-(x, G^-)\, dF(x)$, since $g^-(x, G)$ is increasing in G and any other solution $g(x, G)$ to (7) satisfies $g(x, G) \geq g^-(x, G)$. Thus to obtain a contradiction, suppose that $G^- > \int_{\underline{x}}^{\bar{x}} g^-(x, G^-)\, dF(x)$. Since $\int_{\underline{x}}^{\bar{x}} g^-(x, G)\, dF(x)$ is increasing in G and $\int_{\underline{x}}^{\bar{x}} g^-(x, 0)\, dF(x) \geq 0$, by Tarski's Fixed Point Theorem (e.g., Theorem 2.5.1 in Topkis, 1998), there exists $G' \in [0, G^-)$ such that $G' = \int_{\underline{x}}^{\bar{x}} g^-(x, G')\, dF(x)$, yielding a contradiction.

Next, suppose G^+ is the greatest fixed point of $\phi(G)$. Similarly $G^+ \leq \int_{\underline{x}}^{\bar{x}} g^+(x, G^+)\, dF(x)$, thus to obtain a contradiction, suppose that $G^+ < \int_{\underline{x}}^{\bar{x}} g^+(x, G^+)\, dF(x)$. Since $\lim_{g \to \infty} \chi'(g) = \infty$ and $a \leq 1$, there exists $\bar{G}$ such that $\bar{G} > \int_{\underline{x}}^{\bar{x}} g^+\left(x, \bar{G}\right) dF(x)$. So, since $\int_{\underline{x}}^{\bar{x}} g^+(x, G)\, dF(x)$ is increasing in G, again by Tarski's Fixed Point Theorem, there exists $G' \in \left(G^+, \bar{G}\right)$ such that $G' = \int_{\underline{x}}^{\bar{x}} g^+(x, G')\, dF(x)$, yielding another contradiction and completing the proof of the lemma. ■

Lemma 2 combined with supermodularity of (7) in (a, g) implies that corresponding to the smallest [greatest] fixed point of $\phi(G)$, G^- $[G^+]$, will be the smallest [greatest] aggregate output Q^- $[Q^+]$ corresponding to any fixed point of $\phi(G)$. Lemma 1 then implies that if G_0 is the smallest [greatest] equilibrium value of G and Q_0 is the smallest [greatest] equilibrium value of Q, then there exists an equilibrium with $Q = Q_0$ and $G = G_0$. It is then immediate from (7) that $\phi(G, \tilde{u}, \gamma, \eta)$ is increasing in $\bar{P}$ and γ and decreasing in $\tilde{u}$ and η, for all G. We can also obtain comparative static results with respect to changes in the distribution of productivity, F. For this purpose, throughout the rest of the paper, we use the first-order stochastic dominance order on the distributions $F(\cdot)$ (recall that $F'(x)$ first-order stochastic dominates $F(x)$ if $F'(x) \leq F(x)$ for all x), and use "z is increasing in $F(\cdot)$" as a shorthand for "an increase in $F(\cdot)$ in the first-order stochastic dominance order implies an increase in z"). Given this convention, it is also straightforward that $\phi(G, \tilde{u}, \gamma, \eta)$ is increasing in $F(\cdot)$. Furthermore, by (7) and the fact that $\bar{u}$ is decreasing in G, $\phi(G, \tilde{u}, \gamma, \eta)$ is increasing in G.

Proposition 3 *Suppose $P(\cdot) \equiv \bar{P}$. Then an equilibrium exists and the set of equilibria is a lattice. Furthermore, the smallest and greatest equilibrium aggregates (Q, G) are increasing in $\bar{P}$, γ, and $F(\cdot)$, and decreasing in $\tilde{u}$ and η.*

Proof. As noted in the proof of Lemma 2, $\int_{\underline{x}}^{x^*} g^-(x, 0)\, dF(x) \geq 0$ and there exists $\bar{G}$ such that $\bar{G} > \int_{x^*}^{\bar{x}} g^+(x, \bar{G})\, dF(x)$. This implies that the smallest value of $\phi(0)$ is above the 45° line and the greatest value of $\phi(\bar{G})$ is below the 45° line. Moreover, ϕ is monotone and has well defined smallest and greatest values. Thus, by Tarski's Fixed Point Theorem, it has an intersection with the 45° line between 0 and $\bar{G}$, and from Lemma 1 this implies that an equilibrium exists; and the set of intersections (and thus the set of equilibria) is a lattice. By Lemma 2 and the supermodularity of (7), the smallest and greatest equilibria are well-defined. The argument in the preceding paragraph establishes that $\phi(G)$ is increasing in G, $\bar{P}$, γ, and $F(\cdot)$, and decreasing in $\tilde{u}$ and η, for all G. Since $\phi(G)$ is increasing, Theorem 2.5.2 in Topkis (1998) then implies that the smallest and greatest fixed points of $\phi(G)$, and thus the smallest and greatest equilibrium aggregates, are also increasing in $\bar{P}$, γ, and $F(\cdot)$, and decreasing in $\tilde{u}$ and η. ∎

Note that Proposition 3 relies only on the supermodularity of (7) and the monotonicity of $(a^-, g^-)(x, Q, G)$ and $(a^+, g^+)(x, Q, G)$ in G.[29] In the converse case where $P(QL)$ may depend on Q, but $\gamma = 0$, so that (7) does not depend on G, an equivalent argument using monotonicity of $(a^-, g^-)(x, Q, G)$ and $(a^+, g^+)(x, Q, G)$ in Q yields another analog of Proposition 2 (proof omitted):

Proposition 4 *Suppose $\gamma = 0$. Then an equilibrium exists, and the set of equilibria is a lattice. Furthermore, the smallest and greatest equilibrium aggregates (Q, G) are increasing in $F(\cdot)$ and decreasing in L, $\tilde{u}$, and η.*

Propositions 3 and 4 show that when only one of Q and G is relevant in (7), an equilibrium exists and its comparative statics are well-behaved—the smallest and greatest equilibrium aggregates are increasing in productivity and the difficulty of leaving the coercive sector, and are decreasing in the outside option in the noncoercive sector and the cost of guns. All of these comparative statics are direct consequences of the comparative statics of an individual producer's problem in Proposition 2.

In addition, as a first answer to our question concerning the relationship between labor scarcity and coercion, Proposition 4 formalizes the labor demand effect on coercion, which implies that labor scarcity leads to greater coercion as suggested by Domar. This is because in Proposition 2 both coercion and effort are increasing in the price P, since greater demand for labor increases both, and Proposition 4 then follows from the fact that P is decreasing in L (notably, this result does not apply in Proposition 3, where $P(\cdot) \equiv \bar{P}$). Intuitively, greater labor scarcity reduces production and increases output price, and via this channel increases demand for labor and encourages coercion by individual producers and in the aggregate.

[29] In particular, Proposition 3 does not exploit the fact that, when $P(\cdot) \equiv \bar{P}$, the game is supermodular.

The comparative statics derived in this subsection do not generally hold when both Q and G are relevant in (7). The next subsection provides one case in which both Q and G are relevant in (7) and analogous results do hold, while subsection 4.3 considers the general case.

4.2 Homogeneous Producers

In this subsection, we simultaneously allow $P(QL)$ to depend on Q and allow γ to be positive, but require that all producers are homogeneous, i.e., they all have the same productivity, $\hat{x}$, and in addition we also assume that $c(\cdot)$ satisfies conditions sufficient to guarantee concavity of (7). Since all producer's have the same productivity in this subsection, we also relax the assumption that all producers with the same productivity choose the same (a, g) pair (see also Appendix C).

To ensure concavity, assume in this subsection only that c is three times differentiable and that, for all a, it satisfies

$$(1-a)\, c'''(a) \geq c''(a). \tag{11}$$

This condition guarantees that the producer's problem (7) is strictly concave. It is a slight weakening of the sufficient condition for concavity of the producer's problem imposed by Chwe.[30] If a is an optimal effort for to producer, then concavity, (7), and (10) implies that $\left(a, (\chi')^{-1}\left(\frac{a}{\eta}\right)\right)$ is the unique solution to (7). Homogeneity of producers and strict concavity of (7) implies that all producers choose the same equilibrium contract (a, g) in any equilibrium. In particular, this implies that $a = Q$ and $g = G$ in any equilibrium. This fact, together with the observation that $u^l < 0$ "generically" in partial equilibrium, implies that $u^l < 0$ "generically" in general equilibrium (see Lemma 7 in Appendix A), and thus we can write, from (10), $G = (\chi')^{-1}(Q/\hat{x}\eta)$. Now define

$$\bar{\phi}(Q, \hat{x}, \gamma, L, \tilde{u}, \eta) \equiv a\left(Q, (\chi')^{-1}\left(\frac{Q}{\hat{x}\eta}\right)\right)\hat{x},$$

so that $\bar{\phi}(Q, \hat{x}, \gamma, L, \tilde{u}, \eta)$ is the unique aggregate level of output consistent with each producer choosing an equilibrium contract given aggregate output Q and the unique level of G consistent with Q. The main role of our assumptions here (homogeneous producers and concavity) is that they guarantee the existence of such a unique level of G, by (10) and Lemma 7. We write $\bar{\phi}(Q)$ for $\bar{\phi}(Q, \hat{x}, \gamma, L, \tilde{u}, \eta)$ when the parameters are obvious.

It is clear that if Q is an equilibrium aggregate level of output, then Q is a fixed point of $\bar{\phi}(Q)$; the converse follows from an argument paralleling the proof of Lemma 1. Furthermore, $\bar{\phi}(Q, \hat{x}, \gamma, L, \tilde{u}, \eta)$ is increasing in γ and decreasing in L, $\tilde{u}$, and η. To see this, note that the comparative statics for γ, L, and $\tilde{u}$ are immediate from (7); the comparative static for η follows because a is decreasing in η and increasing in $G = Q/\hat{x}\eta$. The comparative statics with respect to $\hat{x}$ are ambiguous because a is

[30] To see why it ensures concavity, it is sufficient to differentiate (7) when $w^h > 0$ and $u^l < 0$. Chwe also provides a justification for this apparently ad hoc assumption: define $f(\cdot)$ by $f(-\log(1-a)) \equiv c(a)$, so that $f(\rho)$ is the cost to the worker of ensuring success with probability $1 - e^{-\rho}$; then one can verify that $(1-a)\, c'''(a) \geq 2c''(a)$ if $f'''(\cdot) \geq 0$. Our condition simply weakens this to $(1-a)\, c'''(a) \geq c''(a)$.

increasing in both $\hat{x}$ and $G = Q/\hat{x}\eta$.[31] Finally, since $G = Q/\hat{x}\eta$ in any equilibrium, it is immediate that the set of equilibrium (Q, G) pairs forms a lattice.

Thus far, $\bar{\phi}(Q)$ is similar to $\phi(G)$, defined in Section 4.1. An important difference, however, is that $\bar{\phi}(Q)$ may not be monotone, while monotonicity of $\phi(G)$ was the key to establishing Lemma 2, which showed existence of extremal (Q, G) equilibria. However, this problem can be overcome in the current setting thanks to two features. First, we already know that the set of equilibrium (Q, G) pairs is a lattice. Second, we can appeal to the following lemma, which replaces the monotonicity requirements in Lemma 2.[32]

Lemma 3 *Suppose $F(x,t) : [0, \bar{x}] \times \mathbb{R}_+^M \to [0, \bar{x}]$ is a non-empty, upper hemi-continuous, and convex-valued correspondence that is increasing in t in the weak set order for all x. Then the smallest and greatest fixed points of $F(x,t)$ exist and are increasing in t.*

Proof. See Appendix A. ∎

Proposition 5 *Suppose that all producers are homogeneous and (11) holds. Then, an equilibrium exists, the set of equilibria is a lattice, and the smallest and greatest equilibrium aggregates (Q, G) are increasing in γ and decreasing in L, $\tilde{u}$, and η.*

Proof. $\bar{\phi}(0, \hat{x}, \gamma, L, \tilde{u}, \eta) \geq 0$ and $\bar{\phi}(\hat{x}, \hat{x}, \gamma, L, \tilde{u}, \eta) \leq \hat{x}$, and $\bar{\phi}(Q, \hat{x}, \gamma, L, \tilde{u}, \eta)$ is non-empty, upper hemi-continuous and convex-valued by standard arguments. Therefore, the result follows immediately from the fact that $\bar{\phi}(Q, \hat{x}, \gamma, L, \tilde{u}, \eta)$ is increasing in γ and decreasing in L, $\tilde{u}$, and η, by Lemma 3 ∎

Proposition 5 shows that comparative statics are almost exactly as in Propositions 3 and 4 when producers are homogeneous and (11) holds, i.e., $(1-a)\, c'''(a) \geq c''(a)$. The intuition is that, even though both Q and G may now affect (7), the assumptions here are sufficient to ensure that the set of equilibria is a lattice; in particular, the problem is effectively reduced to one where Q is the only aggregate, with G fixed at $(\chi')^{-1}(Q/\hat{x}\eta)$ (from the first-order condition (10)).

4.3 General Case

In this subsection, we consider the model in Section 2 without further restrictions; thus $P(QL)$ may depend on Q, γ is potentially positive, and producers are heterogeneous. In the text, we restrict attention to pure strategies as specified in Definition 1 to simplify notation. To prove existence of equilibria in this general setting, we need to consider mixed strategies; this is done in Appendix C.

Given a map $g(\cdot) : [\underline{x}, \bar{x}] \to \mathbb{R}_+$, let us define

$$G(g(\cdot)) \equiv \int_{\underline{x}}^{\bar{x}} g(x)\, dF(x).$$

[31] The intuition for this is that, while an increase in $\hat{x}$ increases a for fixed (Q, G), it also decreases G for fixed Q, so the overall effect on a of increasing $\hat{x}$ is ambiguous.

[32] $\bar{\phi}(Q, \hat{x}, \gamma, L, \tilde{u}, \eta)$ is a *function* of Q, not a correspondence, so the proof of Proposition 5 relies only on the fact that Lemma 3 holds for continuous functions. However, the proof for non-empty, upper hemi-continuous, and convex-valued correspondences is no harder than for functions, so we state and prove the more general result here.

That is, $G(g(\cdot))$ is the aggregate level of guns when a producer with productivity x chooses $g(x)$. Next, we define

$$\hat{\phi}(Q,\gamma,L,\tilde{u},\eta) \equiv \left\{ \begin{array}{c} Q' : \text{there exists } (a(\cdot), g(\cdot)) \text{ such that } (a(x), g(x)) \text{ is a solution to (7)} \\ \text{given aggregates } (Q, G(g(\cdot))), \text{ and } Q' = \int_{\underline{x}}^{\bar{x}} a(x)\, x dF(x) \end{array} \right\}.$$

That is, $\hat{\phi}$ maps Q and parameter values to those Q' that are equilibrium levels of output in the modified model where price is fixed at $P(QL)$. By Lemma 1, the set of fixed points of $\hat{\phi}$ coincides with the set of equilibrium values of Q.

Lemma 4 *The smallest and greatest elements of $\hat{\phi}(Q,\gamma,L,\tilde{u},\eta)$ are increasing in $F(\cdot)$ and γ, and decreasing in Q, L, $\tilde{u}$, and η.*

Proof. From Proposition 3, the smallest and greatest equilibrium aggregates (Q', G') are increasing in $\bar{P}$, $F(\cdot)$, and γ, and decreasing in $\tilde{u}$ and η, when price is fixed at $\bar{P}$. Since the smallest and greatest elements of $\hat{\phi}(Q,\gamma,L,\tilde{u},\eta)$ are the smallest and greatest equilibrium levels of Q' when price is fixed at $P(QL)$, the results for $F(\cdot)$, γ, $\tilde{u}$, and η follow immediately from Proposition 3, and the results for Q and L follow from Proposition 3 combined with the fact that $P(QL)$ is decreasing. ■

The next proposition shows that the comparative static results obtained so far continue to apply when we focus on the smallest and greatest equilibrium values of output. Note, however, that this proposition does not state that the set of equilibrium values of (Q, G) is a lattice.

Proposition 6 *The smallest and greatest equilibrium values of Q are increasing in $F(\cdot)$ and γ, and decreasing in L, $\tilde{u}$, and η.*

Proof. From Lemma 4, greater $F(\cdot)$ or γ, or lower L, $\tilde{u}$, or η shifts $\hat{\phi}(Q,\gamma,L,\tilde{u},\eta)$ up for all Q. Since $\hat{\phi}(Q,\gamma,L,\tilde{u},\eta)$ is monotone in Q, Theorem 2.5.2 in Topkis (1998) implies that the smallest and greatest fixed points of $\hat{\phi}(Q,\gamma,L,\tilde{u},\eta)$ are increasing in $F(\cdot)$ and γ, and decreasing in L, $\tilde{u}$, and η. ■

Next we can obtain the same results for the smallest and the greatest equilibrium values of G by interchanging the roles of Q and G (proof omitted).

Proposition 7 *The smallest and greatest equilibrium values of G are increasing in $F(\cdot)$ and γ, and decreasing in L, $\tilde{u}$, and η.*

Therefore, whenever we focus on the smallest and greatest equilibrium values of either aggregate, there are unambiguous comparative statics for that aggregate. Intuitively, this is because whenever we take one of the two aggregates as a "parameter," the logic in subsection 4.1 applies and ensures monotonicity of the producers' best responses. Nevertheless, this is not sufficient to give us the most general comparative static results, because the set of equilibrium values of (Q, G) may not be a lattice; and thus one cannot state these results for the smallest and greatest values of the pair of equilibrium aggregates (Q, G).

Finally, the other relevant statement missing from Propositions 6 and 7 concerns the existence of an equilibrium. As noted above, we return to this issue in Appendix C, where we use the distributional approach to equilibrium in games with a continuum of players, originally developed by Mas-Colell (1984), to establish existence of mixed-strategy equilibria, and also generalize Propositions 6 and 7 to mixed strategy equilibria.

4.4 Endogenous Wages in the Outside Sector

We have so far assumed that the wage that an agent receives in the noncoercive (outside) sector, $\tilde{u}$, is constant. This assumption, though convenient, removes a potential general equilibrium effect of labor scarcity: when labor becomes more scarce, this not only changes its marginal product in the coercive sector, but also in the noncoercive sector. This is one interpretation of the neo-Malthusian theories of the decline of feudalism (e.g., Habakkuk, 1958, Postan, 1973, 2002, Leroy Ladurie, 1974, North and Thomas, 1976), where the mechanism via which the fall in population triggered the decline of feudalism may have been the higher wages in the cities resulting from the overall labor scarcity in the economy. We now extend our model to incorporate this (general equilibrium) outside option.

Outside options of coerced agents should naturally be related to opportunities, and thus to the level of employment, in the noncoercive sector. We now introduce this feature and write this outside option as $\tilde{u}(L_N)$, where L_N denotes employment in the noncoercive sector. We then endogenize L_N in the following natural way: suppose that total population is equal to $\tilde{L} \leq 1$, and suppose also that each worker initially has probability $1-\tilde{\gamma}$ of matching with the noncoercive sector, which could correspond to "free cities," and has probability $\tilde{\gamma}$ of matching with a randomly drawn coercive producer. If a worker separates from her employer, she then faces probability γ of matching with a coercive producer and probability $1-\gamma$ of escaping to the noncoercive sector as before.[33] This implies that, after the first round of matching, the environment is identical to our benchmark economy, with $L = \tilde{\gamma}\tilde{L}$ agents matched with coercive producers. Our analysis so far implies that, provided that Assumption 1 is satisfied, the participation constraint of all of agents matched with coercive producers will be hold (as equality), so no agent will leave her coercive employment relationship. Regardless of the value of $\tilde{u}$, the remaining $(1-\tilde{\gamma})\,\tilde{L}$ agents are in the noncoercive sector after the first round of matching, and they will also not leave for a second round of matching (since this could result in matching with a coercive producer, giving them necessarily lower utility than $\tilde{u}$). Therefore, $L_N = (1-\tilde{\gamma})\,\tilde{L}$ is total employment in the noncoercive sector. Suppose that the noncoercive sector has access to a technology of production $H_N\,(L_N)$ with everywhere positive marginal product of labor but diminishing returns, and that $H_N'\,(\cdot) > 0$ and $H_N''\,(\cdot) < 0$. We also assume that workers in the noncoercive sector are paid their marginal product, so that $\tilde{u}(L_N) = H_N'\,(L_N)$. Since $L_N = (1-\tilde{\gamma})\tilde{L}$, the outside option in the noncoercive sector, $\tilde{u}((1-\tilde{\gamma})\tilde{L})$, is strictly decreasing in L_N and thus in $\tilde{L}$. To highlight the new mechanism at work here, let us finally focus on the case where $P\,(\cdot) \equiv \bar{P}$, so that the Domar channel

[33] We allow for two different matching probabilities $\tilde{\gamma}$ and γ (in the first and second rounds of matching) so as to separate the effects of the two in the comparative statics.

highlighted in the previous three subsections is turned off. The next proposition characterizes the equilibrium and provides the main comparative statics.[34]

Proposition 8 *Consider the modified model presented in this subsection and assume that $P(\cdot) \equiv \bar{P}$. Then, an equilibrium exists and the set of equilibria is a lattice. Labor scarcity, corresponding to a decline in $\tilde{L}$, reduces coercion. More formally, a decline in $\tilde{L}$ reduces the smallest and greatest equilibrium aggregates (Q, G). The remaining comparative statics in Proposition 3 continue to apply, and thus the smallest and greatest equilibrium aggregates (Q, G) are increasing in $\bar{P}$, γ, and $F(\cdot)$, and decreasing in $\tilde{u}((1-\tilde{\gamma})\tilde{L})$ and η.*

Proof. $\tilde{L}$ enters the extended model only through $\tilde{u}((1-\tilde{\gamma})\tilde{L})$, so a decline in $\tilde{L}$ here is identical to an increase in $\tilde{u}$ in the model of subsection 4.1 with $P(\cdot) \equiv \bar{P}$. Then, from Proposition 3, an equilibrium exists and the smallest and greatest equilibrium aggregates (Q, G) are decreasing in $\tilde{u}((1-\tilde{\gamma})\tilde{L})$, so are increasing in $\tilde{L}$. Also, again from Proposition 3, the smallest and greatest equilibrium aggregates (Q, G) are increasing in $\bar{P}$, γ, and $F(\cdot)$, and decreasing in $\tilde{u}$ and η. ■

This proposition shows how labor scarcity can lead to lower coercion when it raises the outside option of workers. The most relevant application of this proposition is to situations in which labor scarcity increases the marginal product of workers in noncoercive sectors, such as in the "free cities" during feudal times or in the non-agricultural activities in the context of coercive agricultural labor relations. When this happens, labor scarcity raises the outside option of coerced agents and creates a mechanism towards lower coercion.

Note that Proposition 8 clearly illustrates this mechanism, and thus the intuition implicit in neo-Malthusian theories, because the Domar channel highlighted in Proposition 4 is turned off. If we were to relax the assumption that $P(\cdot) \equiv \bar{P}$, then the implications of labor scarcity would be given by the result of two competing effects. In general, this analysis highlights that the predictions concerning whether labor scarcity will encourage or discourage coercion will depend on whether it mainly affects the returns to producers from extracting effort or the outside option of the agents. This observation is important in the context of Brenner's critique of neo-Malthusian theories. Brenner (1976) pointed out that neo-Malthusian theories were inadequate, because in many instances, and most notably in the episode of the so-called "second serfdom," population declines were associated with more—not less—coercive relationships for peasants. Brenner's interpretation was that the neo-Malthusian approaches were fundamentally inadequate because they ignored the essence of feudalism as a set of coercive relationships. Instead, Brenner favored a Marxian theory based on class conflict (though his theory is not fully articulated). Our model is consistent with the spirit of Brenner's objections to the neo-Malthusian theories, since it explicitly models the coercive relationship (though presumably not in the way that Brenner himself would have done). An interesting implication is that our model is potentially

[34] In Proposition 8, the statement that the smallest and greatest equilibrium aggregates (Q, G) are decreasing in $\tilde{u}((1-\tilde{\gamma})\tilde{L})$ means that the greatest and smallest equilibrium aggregates decrease as the function $\tilde{u}(\cdot)$ evaluated at $(1-\tilde{\gamma})\tilde{L}$ increases.

consistent both with the decline of feudalism and the coercive labor relations that it entailed in Western Europe following large declines in population and also with the increase in coercion in Eastern Europe during the "second serfdom" episode: in the latter case, the decline in population was associated with a significant increase in the demand for agricultural goods produced by East European workers, coming from Western Europe, which translates into an increase in P in the context of our model; and it is plausible that the increase in the outside option of East European workers would have been much more muted than their West European counterparts because of the relative paucity and weakness of cities in this area (which in the context of the current model would correspond to higher γ or a lower $\tilde{u}(\cdot)$ function in Eastern Europe than in the West). This discussion illustrates that the general model that allows both for $P(\cdot)$ and $\tilde{u}(\cdot)$ to be non-degenerate functions can lead to significantly richer, and more subtle, comparative statics, and also highlights that the predictions in general depend on whether labor coercion mainly affects the returns to producers from extracting effort or the outside option of the agents.

4.5 Labor Scarcity and the Returns to Investment in Guns

In this subsection, we highlight another general equilibrium mechanism linking labor scarcity to coercion, which will lead to comparative statics similar to those obtained in the previous subsection. The underlying idea is related to Acemoglu, Johnson and Robinson's (2002) argument that labor coercion is more profitable when there is abundant labor to coerce (because when labor is scarce, coercion can exploit "economies of scale"). This channel is absent in our model so far, because each employer employs (matches with) at most one worker and coercion decisions are taken conditional on the match. An alternative timing of events is to assume that producers invest in guns before the matching stage. This minor change in timing introduces the above-mentioned economies of scale effect and implies that investment in guns will be less profitable when producers are relatively unlikely to match with workers, that is, when labor is scarce. As in the previous subsection, to bring out this particular general equilibrium effect, we abstract from Domar's mechanism by assuming that $P(\cdot) \equiv \bar{P}$.

The only difference from our baseline analysis is that producers choose g before they learn whether they are matched with an agent. This implies that they have a two-stage decision problem, whereby they first choose g before matching and then, after matching, they propose a contract to the agent, which, as before, can be summarized by (a, g). Even though this is a two-stage decision problem, there is no loss of generality in formulating it mathematically as producers choosing a and g simultaneously to solve

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} L\left(a\bar{P}x - a\left[(1-a)\,c'(a) + c(a) + \tilde{u} + \frac{\gamma}{1-\gamma}G - g\right]_+ \right.$$
$$\left. - (1-a)\left[-ac'(a) + c(a) + \tilde{u} + \frac{\gamma}{1-\gamma}G - g\right]_+\right) - \eta\chi(g),$$

with the interpretation that a is their planned level of effort to induce if they match with an agent, and where we have again substituted out the incentive compatibility and participation constraints

under Assumption 1. Rewriting this as

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} a\bar{P}x - a\left[(1-a)\,c'(a) + c(a) + \tilde{u} + \frac{\gamma}{1-\gamma}G - g\right]_+ \\ - (1-a)\left[-ac'(a) + c(a) + \tilde{u} + \frac{\gamma}{1-\gamma}G - g\right]_+ - \frac{\eta}{L}\chi(g),$$

we see that changing the timing of the model by requiring producers to choose g before learning whether they are matched with an agent is formally identical to replacing the cost of guns η with η/L. This implies that the analysis of the set of equilibria is identical to that in Proposition 3, and thus we immediately obtain the following result (proof omitted):

Proposition 9 *Consider the modified model presented in this subsection and assume that $P(\cdot) \equiv \bar{P}$. Then, an equilibrium exists and the set of equilibria is a lattice. Labor scarcity reduces coercion, that is, a decline in L reduces the smallest and greatest equilibrium aggregates (Q, G). Moreover, the smallest and greatest equilibrium aggregates (Q, G) are increasing in $\bar{P}$, γ, and $F(\cdot)$, and decreasing in $\tilde{u}$ and η.*

Therefore, this proposition formalizes another channel via which labor abundance can encourage coercion, and thus complements Proposition 8 in the previous subsection. As in the previous subsection, if we relax the assumption that $P(\cdot) \equiv \bar{P}$, the implications of labor scarcity for coercion will be given by competing effects.

4.6 Welfare Analysis in General Equilibrium

We now revisit the welfare results from Section 3.3, which showed that making coercion cheaper, despite leading to higher effort, reduces utilitarian social welfare. We first show that this conclusion still holds in general equilibrium. We then briefly investigate the implications of "slave trade," where coerced workers can be sold from one producer to another. Finally, we show that coercion in general equilibrium can lead to Pareto dominated outcomes.

To understand the general equilibrium welfare implications of coercion, first note that extremal equilibrium values of Q and G are decreasing in η (from Propositions 6 and 7, or from Propositions 3-5, when they apply). Second, we can observe that Corollary 4—which shows that any equilibrium contract involves greater than first-best effort—continues to hold with endogenous Q and G, replacing "any equilibrium contract" with "the lower bound on equilibrium Q;" this follows from an argument similar to the proof of Corollary 4.

Most importantly, we can also establish that the presence of coercion reduces (utilitarian) social welfare for each matched producer-worker pair. Again from Proposition 6 (or from Propositions 3-5), the smallest and greatest equilibrium values of Q, Q^- and Q^+, are decreasing in η. Therefore, taking η to infinity, which is equivalent to setting $g = 0$ exogenously, reduces the smallest and greatest equilibrium values of Q to $Q^{-\prime} \leq Q^-$ and $Q^{+\prime} \leq Q^+$. Let G^- be the smallest equilibrium value of G.

With coercion, the lower bound on equilibrium social welfare of a matched producer with productivity x and a corresponding matched agent is at most

$$
\begin{aligned}
& a(x) P\left(Q^{+} L\right) x - a(x)(1-a(x)) c'(a) - a(x) c(a(x)) + (1-a(x)) \tilde{u} \\
& - (1-a(x)) \frac{\gamma}{1-\gamma} G^{-} - (1-a(x)) g(x) - \eta \chi(g(x)) \\
< \; & a(x) P\left(Q^{+} L\right) x - a(x)(1-a(x)) c'(a(x)) - a(x) c(a(x)) + (1-a(x)) \tilde{u} \\
\leq \; & \max_{\tilde{a} \in [0,1]} \tilde{a} P\left(Q^{+} L\right) x - \tilde{a}(1-\tilde{a}) c'(\tilde{a}) - \tilde{a} c(\tilde{a}) + (1-\tilde{a}) \tilde{u} \\
\leq \; & \max_{\tilde{a} \in [0,1]} \tilde{a} P\left(Q^{+\prime} L\right) x - \tilde{a}(1-\tilde{a}) c'(\tilde{a}) - \tilde{a} c(\tilde{a}) + (1-\tilde{a}) \tilde{u},
\end{aligned}
$$

where the last line is a lower bound on total welfare of the same pair of producer and agent in any equilibrium with $g \equiv 0$ (since, from Proposition 1, $w^h \geq 0$ when $g = 0$). Similarly, the upper bound on total equilibrium welfare of the pair is at most

$$
\begin{aligned}
& a(x) P\left(Q^{-} L\right) x - a(x)(1-a(x)) c'(a(x)) - a(x) c(a(x)) + (1-a(x)) \tilde{u} \\
& - (1-a(x)) \frac{\gamma}{1-\gamma} G^{-} - (1-a(x)) g(x) - \eta \chi(g(x)) \\
< \; & a(x) P\left(Q^{-} L\right) x - a(x)(1-a(x)) c'(a(x)) - a(x) c(a(x)) + (1-a(x)) \tilde{u} \\
\leq \; & \max_{\tilde{a} \in [0,1]} \tilde{a} P\left(Q^{-} L\right) x - \tilde{a}(1-\tilde{a}) c'(\tilde{a}) - \tilde{a} c(\tilde{a}) + (1-\tilde{a}) \tilde{u} \\
\leq \; & \max_{\tilde{a} \in [0,1]} \tilde{a} P\left(Q^{-\prime} L\right) x - \tilde{a}(1-\tilde{a}) c'(\tilde{a}) - \tilde{a} c(\tilde{a}) + (1-\tilde{a}) \tilde{u},
\end{aligned}
$$

where the last line is now the upper bound on equilibrium total welfare of the pair when $g \equiv 0$. This establishes that the upper [lower] bound on equilibrium total welfare for each pair is less than the upper [lower] bound on welfare when $g \equiv 0$. Aggregating across all pairs implies that upper [lower] bound on total welfare in the society is less than the upper [lower] bound on total welfare without coercion, which generalizes our partial equilibrium result (Corollary 5). We summarize this discussion in the following corollary (proof in the text):

Corollary 8 *Social welfare in any equilibrium under coercion ($\eta < \infty$) is strictly lower than social welfare in any equilibrium under no coercion.*

We next turn to the issue of "slave trade". We have so far assumed that agents are matched to coercive producers, and contracting and production take place between this matched pair. This setup thus allows labor coercion, but rules out "slave trade," whereby agents subject to coercion are sold from one producer to another. We next investigate how this type of trading affects the equilibrium and its welfare properties.[35]

[35] Note that this is different from the effect of "slave trade" that would bring additional slaves from the outside, as the Atlantic slave trade did until its abolition in 1807. We show that slave trade reduces worker welfare even if one ignores its effect on the number of coerced workers (which is outside our model).

The answer to this question is straightforward to develop given our analysis so far. Recall that so far matched producers are a random selection from the population of producers, and thus their productivity distribution is given by $F(x)$. Since $L < 1$, some producers are left unmatched. Now suppose that, before the investment in guns, agents can be bought and sold among producers. Since more productive (higher x) producers have a greater willingness to pay for coerced agents, this trading will ensure that all of the agents will be employed by the most productive L producers. Consequently, the distribution of productivity among matched producers will be $\tilde{F}(x) = 0$ for all $x < x_{1-L}$ and $\tilde{F}(x) = F(x)/L$ for all $x \geq x_{1-L}$, where x_{1-L} is the productivity of the producer at the $(1-L)$th percentile of the productivity distribution. This implies that slave trade is equivalent to a first-order stochastically dominating shift in the distribution of productivity in the context of our model, and the analysis so far implies that this will raise coercion.

Now using this observation and the fact that agents matched to more productive producers have lower welfare (Corollary 3), the next result shows that slave trade, even conditional on coercion, increases the amount of coercion in equilibrium and reduces agent welfare. It may even reduce overall welfare, despite the fact that agents are now allocated to the most productive producers.

Proposition 10 *Introducing slave trade in the baseline model increases coercion (G) and reduces agent welfare. More formally, the smallest and the greatest equilibrium levels of coercion [average agent welfare] under slave trade are greater [smaller] than the smallest and the greatest equilibrium levels of coercion [average agent welfare] under no slave trade. In addition, social welfare may decline under slave trade.*

Proof. Most of the proposition follows from the preceding discussion along with Proposition 7. The only part to be proved is that overall welfare may decline under slave trade. We illustrate this using a simple example, with two types of producers (see Appendix C for analogs of our earlier results in the general case where the distribution F can have mass points). Let $L = 1/2$, $P(Q) = 1$ for all Q, $\gamma = 0$, $c(a) = 1/(1-a)$, $\chi(g) = g^2/2$, and $x = x_1$ or $x_2 > x_1$ with equal probability. Suppose that Assumption 1 is satisfied, i.e., $Px_1 > \tilde{u} + c'(0)$. Notably, there are no general equilibrium interactions in this example, so the only effect of the slave trade is shifting mass $1/4$ workers from producers with productivity x_1 to producers with productivity x_2. Therefore, it suffices to show that social welfare in a producer-worker pair with productivity x_1 is higher than in a producer-worker pair with productivity x_2. Using the fact that $g = a/\eta$, the maximization problem (7) for a producer with productivity x can be written as

$$\max_{a\in[0,1]} xa - a\left(\frac{2}{1-a} + \tilde{u} - \frac{a}{\eta}\right) - \frac{1}{2}\frac{a^2}{\eta}.$$

Now assuming in addition that $\tilde{u} = 1$, $\eta = 2$, $x_1 = 3.1$ and $x_2 = 3.2$ (which clearly satisfy Assumption 1), we obtain that $a(x_1) \simeq 0.028$, which yields producer welfare of approximately 0.0014, and worker and total (utilitarian) welfare of approximately 0.99. In contrast, $a(x_2) \simeq 0.052$, which yields producer welfare of 0.005, worker welfare of approximately 0.97, and total welfare of approximately 0.98, which

is lower than the corresponding social welfare for $x_1 = 3.1$. This illustrates the possibility of slave trade reducing total social welfare and completes the proof of the proposition. ■

An implication of Corollary 10, highlighted by the example used in the proof, is that an increase in productivity can reduce total welfare because of its effect on coercion. In particular, increasing productivity x has two offsetting effects on social welfare. First, fixing a, increasing x by Δ increases expected output by Δa (a first-order positive effect). Second, increasing x increases a and thus g. The increase in g has a first-order negative effect on worker welfare, but no first-order effect on producer welfare by the Envelope Theorem. This reasoning also suggests that, as in the example in the proof, an increase in productivity is more likely to reduce social welfare when a is small.

Finally, in partial equilibrium, even though coercion reduces the welfare of the producer and the agent combined, it naturally increases the welfare of the producer (who chooses the coercion). In general equilibrium, there is a stronger sense in which coercion may be "inefficient": an equilibrium with coercion may be Pareto dominated by an equilibrium in which coercion is banned, in the sense that both producers and agents are worse off with coercion. When this happens, potentially coercive producers may become in favor of abolishing slavery or limiting coercion. The following result provides conditions under which banning coercion must be Pareto improving if $P(\cdot)$ declines sufficiently steeply.[36]

Proposition 11 *Suppose that (i) $P(QL)$ satisfies $P(QL) \geq \bar{P}$ if $Q \leq \underline{Q}$ and $P(QL) \leq \underline{P}$ if $Q \geq \bar{Q}$; (ii) if coercion is banned ($\eta = \infty$), $P = P(0)$, and $\bar{u} = \tilde{u}$, then for any x all equilibrium contracts $(a, g)(x, P(0), \tilde{u})$ have $a(x, P(0), \tilde{u}) \leq \underline{Q}$; (iii) if coercion is allowed ($\eta < \infty$), $P = P(L)$, and $\bar{u} = \tilde{u}$, then for any x all equilibrium contracts $(a, g)(x, P(L), \tilde{u})$ have $a(x, P(L), \tilde{u}) \geq \bar{Q}$. Then there exists $P^* > 0$ such that, if $\underline{P} < P^*$, then any equilibrium with coercion banned Pareto dominates any equilibrium with coercion allowed.*

Proof. If coercion is banned, then $\bar{u} = \tilde{u}$ in any equilibrium, so for any x the highest candidate equilibrium contract is $(a, g)(x, P(0), \tilde{u})$. If $a(x, P(0), \tilde{u}) \leq \underline{Q}$ in each of these, then $a(x, P, \bar{u}) \leq \underline{Q}$ for all x in any equilibrium, which implies that $P \geq \bar{P}$ in any equilibrium.

If coercion is allowed, then for any x the lowest candidate equilibrium contract is $(a, g)(x, P(L), \tilde{u})$. If $a(x, P(0), \tilde{u}) \geq \bar{Q}$ in each of these, then $a(x, P, \bar{u}) \geq \bar{Q}$ for all x in any equilibrium, which implies that $P \leq \underline{P}$ in any equilibrium.

If $P \leq \underline{P}$, then a producer's expected payoff is no more than $\underline{P}$. So as $\underline{P} \to 0$, a producer's expected payoff in any equilibrium with coercion allowed goes to 0, while a producer's expected payoff in any equilibrium with coercion banned remains constant, completing the proof. ■

[36] A numerical example illustrating the conclusion of this proposition is available from the authors upon request.

5 Extensions

In this section, we discuss several extensions of the basic model presented so far. To reduce notation, in the remainder of the text we focus on the partial equilibrium results.

5.1 Ex Ante Investments and Effort- vs. Skill-Intensive Labor

A natural conjecture would be that, in addition to the inefficiencies associated with coercion identified above, coercive relationships discourage ex ante investments by workers. For example, one could use the benchmark model of relationship specific investments introduced in Grossman and Hart (1986), equate "coercion" with the ex post bargaining power of the employer, and thus conclude that coercion should discourage investments by the worker, while potentially encouraging those by the producer (employer). Our model highlights that the effect of coercion on investments is more complex. In particular, coercion will encourage agents to undertake investments that increase their outside options, while at the same time giving them incentives to *reduce* (if they can) their productivity within the relationship. This is because, as shown in Section 3, greater outside options increase the agent's payoff more than one-for one (by also reducing coercion), while greater productivity inside the coercive relationship reduces the agent's payoff (by increasing coercion). Conversely, the producer will undertake those investments that increase productivity more than the outside option of the agent.[37] One implication of this result is that the presence of coercion may encourage agents to invest in their general human capital, though for very different reasons than the standard Becker approach would suggest. Coercion also discourages them from investing in relationship-specific human capital; in fact, it gives them incentives to sabotage their producer's productive assets.

We model these issues by adding an interim stage to our game between matching and the investment in guns, g. At this stage, matched agents and producers make investment decisions, denoted by i and I, respectively. For simplicity, we will analyze the case in which such investment opportunities are available only to one party (either the agent or the producer).[38] Investments potentially affect both productivity x within the relationship and the worker's outside option $\bar{u}$, which we now write as either $x(i)$ and $\bar{u}(i)$, or $x(I)$ and $\bar{u}(I)$, depending on which side makes the investment. Suppose that investment i has costs $\zeta(i)$ for the agent, while investment I costs a producer $\tilde{\zeta}(I)$. We assume that both cost functions are increasing and convex. We also further simplify the discussion throughout by focusing on the generic set of parameters for which $u^l < 0$ in any equilibrium contract, so that we can replace (7) by (8) and ensure differentiability of g in parameters, and by imposing two additional assumptions: (1) The equilibrium contract is unique for all x. (2) $sign(Px'(i) - \bar{u}'(i))$ and $sign(Px'(I) - \bar{u}'(I))$ do not depend on i and I, i.e., each investment always has a larger effect on either Px or for $\bar{u}$, and thus either $sign(Px'(i) - \bar{u}'(i)) < 0$ for all i or $sign(Px'(i) - \bar{u}'(i)) > 0$ for

[37] Indeed, if the producer's ex ante investment decreases the agent's outside option and has no effect on productivity, it is simply a relabeling of guns.

[38] This implies that we are abstracting from indirect effects resulting from the interaction of investments by agents and producers.

all i, and likewise for $sign\left(Px'(I) - \bar{u}'(I)\right)$. These last two assumptions are imposed only to facilitate the exposition. If the first assumption is relaxed, one has to take care of the situations in which a change in the investment of the agent can induce a change in equilibrium contract, though this does not affect any of our comparative statics.[39] Thus this assumption is without loss of much generality. The second assumption enables us to clearly separate two different cases, which will have different implications.

Let us first analyze the situation in which only agents have investment opportunities. As a benchmark, note that if there is no coercion (i.e., $\eta = \infty$), a matched worker anticipates receiving expected utility $\bar{u}(i) - \zeta(i)$ after choosing investment i, and therefore chooses i to solve

$$\max_{i \in \mathbb{R}_+} \bar{u}(i) - \zeta(i). \tag{12}$$

Returning to the analysis in Section 3, it is clear that when there is the possibility of coercion (recall that guns, g, is chosen after the agent's investment, i), the agent will receive utility $\bar{u}(i) - g(i) - \zeta(i)$, and she therefore chooses i to solve

$$\max_{i \in \mathbb{R}_+} \bar{u}(i) - g(i) - \zeta(i). \tag{13}$$

To characterize the solution to this program, we need to determine how the choice of guns by the producer responds to changes in i, i.e., the form of the function $g(i)$. Equation (10) implies that $g(i) = (\chi')^{-1}(a(i)/\eta)$; under the assumption that the equilibrium contract is unique, we can differentiate this to obtain

$$g'(i) = \frac{a'(i)}{\eta \chi''(g(i))}. \tag{14}$$

Next, note that the producer's expected profit in an equilibrium contract may be written as

$$a\left(Px(i) - \bar{u}(i)\right) - a(1-a)c'(a) - ac(a) + ag - \eta\chi(g). \tag{15}$$

Therefore, $sign\left(a'(i)\right) = sign\left(Px'(i) - \bar{u}'(i)\right)$, and thus

$$sign\left(g'(i)\right) = sign\left(Px'(i) - \bar{u}'(i)\right).$$

Combining this with (12) and (13) then immediately yields the following result (proof in the text):

[39] In particular, if two (or more) equilibrium contracts existed, the producer could implicitly threaten the agent with a switch from one equilibrium to the other. We do not find such threats plausible, since they rely on the indifference of the producer between several options (such switches can never be strict best responses). Moreover, they can only arise when there exist multiple equilibrium contracts for an interval of values of the agent's investments, which cannot be the case in our framework.

In any case, it can be shown that even if there are multiple equilibrium contracts, all of the results in this subsection continue to hold, though the analysis needs to be more involved, because we cannot use the differentiability of equilibrium effort and coercion ($a(i)$ and $g(i)$). Instead, we would first have to show that an equilibrium cannot be at the point of multiplicity if the producer used the multiplicity as an implicit threat, because a small change in i would have a noninfinitesimal effect on agent welfare, and then given this result, we could develop the same argument as in the text.

Proposition 12 *Equilibrium investment by the agent, i^C, under coercion, given by the solution to (13), is greater than the noncoercion investment, i^N, given by the solution to (12), if and only if*

$$sign\left(Px'(i) - \bar{u}'(i)\right) < 0.$$

Proposition 12 implies that under coercion agents will underinvest (relative to noncoercion) in tasks that increase their within-relationship productivity relative to their outside option. The intuition is straightforward: when the agent becomes more productive, the producer chooses a contract that induces greater effort, which also comes with greater coercion. This reduces agent welfare, so, all else equal, the agent would like to avoid this. Counteracting this effect is the fact that investment may decrease coercion by increasing the agent's outside option. Thus, if we focus on investments that primarily affect the outside option of the agent, coercion increases investment. Conversely, when investments are most relevant for within-relationship productivity, the possibility of coercion reduces investments.

An interesting implication of Proposition 12 relates to the argument by Fenoaltea (1984), that slavery is often observed in "effort-intensive" tasks, but not in "care-intensive" tasks (see also Engerman and Sokoloff, 1998). Fenoaltea attributes this association to the psychological difficulty of using punishments to motivate care. Our result provides an alternative explanation, under the assumption that "care-intensive" tasks are those where relationship-specific investment by the worker are more important—in this interpretation a corresponds to "effort," while i is associated with "care," and we have in mind tasks where $Px'(\cdot) - \bar{u}'(\cdot) > 0$.[40]

Another implication is that if the agent had the means to "sabotage" the productivity of her employer, she would like to do so, because by reducing this productivity, she would reduce both the amount of effort that she is induced to exert and the degree of coercion that she is subjected to.

Next, consider the situation where only producers undertake investments. Without coercion, a producer who chooses I and a receives expected payoff

$$a\left(Px(I) - \bar{u}(I)\right) - a(1-a)c'(a) - ac(a) - \tilde{\zeta}(I), \tag{16}$$

while with coercion he receives expected payoff

$$a\left(Px(I) - \bar{u}(I)\right) - a(1-a)c'(a) - ac(a) + ag - \eta\chi(g) - \tilde{\zeta}(I). \tag{17}$$

Clearly, the producer will choose $I = 0$ if $Px'(I) - \bar{u}'(I) \leq 0$, regardless of whether we allow coercion; this is a version of the standard result in human capital theory that producers never provide general skills training. If, on the other hand, $Px'(I) - \bar{u}'(I) > 0$, then with the same arguments as in Section 3, it can be verified that (17) is supermodular in $(I, a, g, -\eta)$. Now the comparison between producer

[40]This is consistent with Fenoaltea's discussion, which emphasizes the association between "care-intensive" and skill-intensive tasks. For example, he notes that uncoerced galley crews were sometimes used because "the technically superior rowing configuration did require skilled oarsmen" (1984, p. 642), and that "at least the skilled branches of factory production" were care-intensive (p. 654).

investment under coercion and noncoercion can be carried out by noting that (16) is equivalent to (17) as $\eta \to \infty$. Supermodularity of (17) then immediately gives the following result (proof in the text):

Proposition 13 *Equilibrium investment by the producer, I^C, under coercion, given as the solution to (17), is greater than the noncoercion investment, I^{NC}, given as the solution to (16), if and only if*

$$sign\left(Px'(I) - \bar{u}'(I)\right) > 0.$$

Proposition 13 has a similar interpretation to Proposition 12: investment incentives are determined by whether they affect productivity within the relationship or the outside option by more. In contrast to the agent, the producer has greater incentives to invest when relationship-specific productivity increases by more than the outside option of the worker. The general principle here is related to the result in Acemoglu and Pischke (1999) that employers will invest in general human capital when there is "wage distortion," so that these investments increase worker productivity inside the relationship by more than their outside wage.

5.2 Collective Coercion

In many historical episodes, a group of producers exercises coercion collectively, using local or national institutions, law enforcement agencies, or various groups of paramilitaries. In this subsection, we show that our main comparative static results generalize to an environment in which coercion is undertaken collectively by a finite number of (matched) producers. Formally, we assume that there are $N < \infty$ producers and agents matched together, and that the environment is identical to the partial equilibrium model of Section 3 with the sole modification that the level of coercion g faced by agent i is

$$g \equiv \frac{1}{N}\sum_{j=1}^{N} g_j,$$

where g_j is the "contribution" of producer j to the collective coercion technology. Clearly, this specification makes coercion a privately provided public good for the producers.

The following proposition can be established using the same arguments leading to Propositions 1 and 2, and we omit the proof for brevity.

Proposition 14 *Suppose $Px > \bar{u} + c'(0)$. Then in any equilibrium contract each producer i (of type x) chooses $a_i > 0$ and $g_i > 0$, such that*

$$\begin{aligned}(a_i, g_i) \in \arg\max_{(\tilde{a}_i, \tilde{g}_i)\in[0,1]\times\mathbb{R}_+} & Px\tilde{a}_i - \tilde{a}_i\left[(1-\tilde{a}_i)c'(\tilde{a}_i) + c(\tilde{a}_i) + \bar{u} - \frac{1}{N}\sum_{j\neq i}\tilde{g}_j - \frac{1}{N}\tilde{g}_i\right]_+ \\ & -(1-\tilde{a}_i)\left[-\tilde{a}_i c'(\tilde{a}_i) + c(\tilde{a}_i) + \bar{u} - \frac{1}{N}\sum_{j\neq i}\tilde{g}_j - \frac{1}{N}\tilde{g}_i\right]_+ - \eta\chi(\tilde{g}_i), \qquad (18)\end{aligned}$$

with $w^l = p^h = 0$, $w^h = (1-a)c'(a) + c(a) + \bar{u} - g > 0$, *and* $p^l = ac'(a) - c(a) - \bar{u} + g \geq 0$, *where* $g = \frac{1}{N}\sum_{j=1}^{N} g_j$.

Moreover, the set of equilibrium contracts $(a(x), g(x))$ *of a producer of type* x *forms a lattice, with greatest and smallest elements* $(a^+(x), g^+(x))$ *and* $(a^-(x), g^-(x))$. *The extremal equilibrium contracts* $(a^+(x), g^+(x))$ *and* $(a^-(x), g^-(x))$ *are increasing in* x *and* P *and decreasing in* $\bar{u}$ *and* η, *and thus (extremal equilibrium) coercion levels,* g^+ *and* g^-, *are increasing in* P *and* $F(\cdot)$, *and decreasing in* $\bar{u}$ *and* η.

Intuitively, the maximization problem (18) is identical to (7) with $\bar{u} - \frac{1}{N}\sum_{j \neq i} \tilde{g}_j$ replacing $\bar{u}$ and $\chi(N\tilde{g}_i)$ replacing $\chi(\tilde{g}_i)$. This is the reason why all of our comparative statics (Proposition 2) continue to hold with collective coercion (and are stated as part of Proposition 14). However, because coercion is a public good for the producers, free-riding among them implies that less coercion will used in equilibrium than if the producers could solve their collective action problem.

This proposition is useful in highlighting that our analysis can provide the basis of richer theories of coercion, where the collective nature of coercion and its institutional foundations are also explicitly modeled, and our comparative static results are likely to hold in these more general environments.

5.3 Limited Commitment to Payments

A major simplifying assumption of our analysis so far is that the producer can commit to any output-contingent payment-punishment plan (w^y, p^y). There is a natural tension raised by this assumption, however. Contingent payments are often difficult to specify in contracts, which has motivated the relational contracts literature (see, e.g., Levin, 2003). This issue is particularly problematic in our context, since the producer has guns and the capacity to coerce, and can use these to renege on his promises.[41] In particular, a producer with more guns may face greater difficulty in committing to a contract, creating a form of *holdup* problem.

In this subsection, we provide a brief, reduced-form analysis of this *producer holdup* problem. We assume that the producer with guns equal to g can renege on the payments promised to the agent at a cost $h(g)$. For example, this may reflect the fact that the producer needs to threaten or apply violence in order to renege on the promised payment, which will typically be costly. This possibility introduces a natural holdup constraint, whereby if either w^h or w^l exceed $h(g)$, then the producer will exercise the option of using coercion in order to avoid payment. Since this will be perfectly anticipated by the agent, no payment above $h(g)$ is credible or will be observed in equilibrium. Since equilibrium contracts will still involve $w^l \leq w^h$, the relevant constraint is

$$w^h \leq h(g). \tag{H}$$

[41] Interestingly, historical coercive labor "contracts" appear to have been have "informal" or "relational," even when coercion was legal (e.g., Fogel and Engerman, 1974). A full analysis of this issue may necessitate an infinitely repeated game between the producer and the agent. Nevertheless, the static problem presented here is important, since we know from the relational contracting literature (especially Levin, 2003), that optimal relational contracts between risk-neutral parties often closely resemble optimal static contracts subject to payment constraints such as (H).

To reflect the fact that a producer with more guns finds it more difficult to commit to paying the agent, we assume throughout this section that

$$h'(g) \leq 0 \text{ for all } g$$

(we also maintain the convenient assumption that $h(\cdot)$ is differentiable).

The rest of the relationship between matched producer and agent is the same as in Section 3, and thus, with analogy to our analysis there, an equilibrium contract is a solution to the maximization of (1) subject to (IC_0), (IR_0), and now (H). If this last constraint is slack, the solution in Section 3 applies.

Here we therefore focus on the case in which (H) is binding, and thus $w^h = h(g)$. In that case, we can combine (H) with (IC_0) and (IR_0). A possible complication is that, in the presence of (H), it is no longer the case that $w^l = 0$ necessarily (because the producer may need to pay, rather than punish, following low output in order to satisfy the participation constraint of the agent without paying a wage rate that would violate (H)). Nevertheless, a similar analysis to that in Section 3 can be developed. First, note that the first-order approach still applies and thus, using the fact that $w^h = h(g) > 0$, we can write a version of (IC_1) as

$$h(g) - \left(w^l - p^l\right) = c'(a).$$

This enables us to write (IR_0) as

$$h(g) + g \geq \bar{u} + (1-a)\,c'(a) + c(a), \tag{19}$$

which defines the set of feasible (a, g), and $w^l = [h(g) - c'(a)]_+$. A useful observation is that if $h'(\cdot) > -1$ (i.e., $h'(g) > -1$ for all g), then the left-hand side is increasing in g, and thus (19) forms a lattice (or more appropriately a sublattice of $\mathbb{R}^2_+$). Conversely, if $h'(\cdot) < -1$, then the left-hand side is decreasing in g, and (19) forms a lattice with the reverse order on g (i.e., in $(a, -g)$).

Substituting $w^h = h(g)$ also into (1) and using the fact that $w^l = [c'(a) - h(g)]_+$, we obtain that an equilibrium contract is now a solution to

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} a\,(Px - h(g)) - (1-a)\left[h(g) - c'(a)\right]_+ - \eta\chi(g) \tag{20}$$

subject to (19).

It is straightforward to verify that the maximand in (20) is still supermodular in (a, g), by our assumption that $h'(\cdot) \leq 0$. When $h'(\cdot) > -1$, (19) forms a lattice, so equilibrium contracts are still solutions to a supermodular maximization problem in (a, g). This analysis therefore establishes the following (proof in the text):[42]

Proposition 15 *Consider the problem with producer holdup and suppose that (H) binds at every equilibrium contract and that $h'(\cdot) > -1$. Then the set of equilibrium contracts is a lattice with the usual order on $\mathbb{R}^2_+$, and the weak version of the comparative static results of Proposition 2 apply.*

[42] Note that Proposition 15 only establishes weak comparative statics, while Proposition 2 established strict comparative statics.

Intuitively, with holdup, an increase in g creates two effects. The first is a direct effect on the worker's participation constraint as in our baseline model of Section 3. The second is an indirect effect working through the fact that higher g now limits the amounts that the worker can be paid when successful. When $h'(\cdot) > -1$, the direct effect outweighs the indirect effect, and thus increasing g continues to relax (19); consequently, the comparative statics of Section 3 still apply.

The situation is different when $h'(\cdot) < -1$, because, as noted above, the constraint (19) now forms a lattice with the reverse order on g, implying that higher g will typically require a lower a. In this case, the comparative static results of Section 3 no longer apply. To see this most clearly, suppose that the equilibrium contract is unique and involves $w^l = h(g) - c'(a) > 0$. In this case, (20) can be written as

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} aPx - h(g) + (1-a)\,c'(a) - \eta\chi(g), \tag{21}$$

subject to (19), and small changes in parameters will lead to small changes in the solution to this maximization problem (which still induces $w^l > 0$). In this case, the objective function is weakly supermodular (or submodular as well as supermodular) and the constraint is a lattice in $(a, -g)$). Therefore, in this case the *opposite* of the comparative statics of Section 3 will hold—in the sense that coercion will be decreasing in x and P, while effort will be increasing in the cost of guns, η. Intuitively, because now $h'(\cdot) < -1$, the left-hand side of (19) is decreasing in g, and thus increasing coercion tightens this constraint—an increase in g necessitates significantly lower wages to satisfy (H) and this more than offsets the direct effect of greater coercion on the agent's participation constraint.

5.4 Interim Participation Constraints

Another implicit assumption of our analysis so far has been that the agent can run away immediately after the match (i.e., after the "contract offer" of the producer), but not subsequently. An alternative perspective would be that the agent could also attempt to run away after observing an unsuccessful outcome, i.e., precisely when she will receive punishment rather than reward. This additional option to run away would introduce one more "IR" or participation constraint for the agent, which we refer to as the *interim participation constraint*. The presence of such an intern participation constraint introduces a potential "useful" role of coercion. Intuitively, the agent may prefer ex ante to be unable to run away after output is realized; this is similar to the logic that operates in models such as Chwe's, where coercion does not affect the ex ante participation constraint but does enable punishment. Interestingly, however, we will see that, under fairly weak assumptions, this effect is dominated by the negative impact of coercion on welfare.

Formally, in this subsection we assume that the agent will run away after the realization of y if her (equilibrium-path) continuation utility is below $\bar{u} - \Phi(g)$, where $\bar{u}$ is the outside option of the agent introduced above and $\Phi(g)$ is an increasing function of guns; the interpretation of this is that the producer can inflict punishment $\Phi(g)$ if the agent runs away after the realization of y, just as

she can inflict punishment g if the agent runs away before the realization of y.[43] This consideration therefore introduces an "interim IR constraint," (IIR), in addition to the "ex ante IR constraint," (IR_0), in Section 2. This implies that both $w^h - p^h$ and $w^l - p^l$ must be greater than $\bar{u} - \Phi(g)$, though naturally only the constraint $w^l - p^l \geq \bar{u} - \Phi(g)$ can be binding in an equilibrium contract. Therefore, the considerations discussed in this subsection introduce the additional constraint

$$w^l - p^l \geq \bar{u} - \Phi(g). \tag{IIR}$$

Now, by an argument similar to that in Section 3, an equilibrium contract is a solution to the maximization of (1) subject to (IC_0), (IR_0), and now (IIR).

Suppose first that $\Phi(g)$ is sufficiently large for all g. It is then clear that (IIR) will always be slack and the producer's problem is identical to that in Section 3, and Proposition 2 applies. In this subsection, we thus suppose instead that $\Phi(g) \leq g$. This case turns out to be tractable because now (IR_0) will be slack. This is shown by the following lemma:

Lemma 5 *Consider the model with the interim participation constraint and suppose that $\Phi(g) \leq g$. Then (IR_0) is slack and (IIR) binds.*

Proof. The first-order approach still applies, so we can replace (IC_0) with (IC_1). By (IIR), $w^l - p^l \geq \bar{u} - \Phi(g)$. Substituting this into (IC_1) gives $w^h - p^h \geq c'(a) + \bar{u} - \Phi(g)$. Combining these inequalities yields

$$\begin{aligned} a\left(w^h - p^h\right) + (1-a)\left(w^l - p^l\right) - c(a) &\geq \bar{u} - \Phi(g) + ac'(a) - c(a) \\ &\geq \bar{u} - \Phi(g) \\ &\geq \bar{u} - g, \end{aligned}$$

where the second inequality follows by convexity of $c(a)$ and the fact that $c(0) = 0$, and the third follows by the assumption that $\Phi(g) \leq g$. This chain of inequalities implies (IR_0). Given that (IR_0) is slack, the fact that increasing p^l relaxes (IC_1) implies that (IIR) must bind. ■

Provided that $\Phi(g) \leq g$, Lemma 5 then allows us to substitute (IC_1) into (IIR), which implies that equilibrium contracts are characterized by

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} aPx - a\left[c'(a) + \bar{u} - \Phi(g)\right]_+ - (1-a)\left[\bar{u} - \Phi(g)\right]_+ - \eta\chi(g). \tag{22}$$

(22) is supermodular in $(a, g, x, P, -\bar{u}, -\eta)$, so comparative statics go in the same direction as in our baseline model, though they may hold with weak rather than strict inequalities:[44]

[43] For example, g may be the pain that the producer can inflict on a worker if she runs away on the first day on the job, while $\Phi(g)$ may be the pain that the producer can inflict on the worker once she has set up a home on the producer's plantation.

[44] If $\bar{u} - h(g) > 0$, a and g are only weakly complementary in the producer's objective (i.e., the problem is both supermodular and submodular). In this case, a change in one of x, P, $\bar{u}$, and η will only affect the smallest and greatest equilibrium contract values of either a or g, leaving the smallest and greatest equilibrium contract values of the other variable constant.

Proposition 16 *If $\Phi(g) \leq g$ for all g, then the greatest and smallest equilibrium contracts $(a^+(x), g^+(x))$ and $(a^-(x), g^-(x))$ are nondecreasing in x and P and nonincreasing in $\bar{u}$ and η.*

The intuition for this is that, regardless of whether (IR_0) or (IIR) are binding, increasing g reduces the amount that the producer must pay the agent after high output is realized, leading to complementarity between effort and coercion as in Section 3.

While the possibility that (IIR) rather than (IR_0) may be binding does not affect our comparative static results, it does suggest that coercion may play a socially useful role. In particular, ex post punishments may be useful in providing incentives to an agent who is subject to limited liability, and (IIR) limits the use of such punishments; one may then conjecture that increasing g may increase social welfare if it relaxes (IIR). We next show that this conjecture is not correct. To see this, note that if (IIR) is binding and coercion is not allowed (i.e., $\eta = \infty$), then the producer's problem becomes

$$\max_{a \in [0,1]} aPx - ac'(a) - a\bar{u} - (1-a)[\bar{u}]_+,$$

and (utilitarian) social welfare is

$$SW^N = \max_{a \in [0,1]} aPx - ac'(a) - a\bar{u} - (1-a)[\bar{u}]_+ + \bar{u};$$

while with coercion the producer's problem is given by (22), and social welfare corresponding to an equilibrium contract involving (a, g) is

$$SW^C = aPx - ac'(a) - a\bar{u} + a\Phi(g) - (1-a)[\bar{u} - \Phi(g)]_+ - \eta\chi(g) + \bar{u} - g.$$

If $\Phi(g) \leq g$ for all g, then it is clear that $SW^N \geq SW^C$, with strict inequality if $g > 0$. Thus, coercion reduces social welfare, if $\Phi(g) \leq g$. Formally, we have the following (proof in text):

Corollary 9 *Suppose that $\Phi(g) \leq g$ for all g, and let SW^C be social welfare corresponding to an equilibrium contract with coercion. Then $SW^N \geq SW^C$, with strict inequality if $g > 0$.*

The intuition for Corollary 9 follows by comparing (7) to (22). Both (7) and (22) are supermodular in (a, g)—because of the term ag in (7) and the term $a\Phi(g)$ (or, if $w^l > 0$, $\Phi(g)$) in (22). Our result in Section 3 that coercion reduces social welfare exploits the fact that coercion enters into worker welfare through the term $-g$, which is always larger in absolute value than ag. If $\Phi(g) \leq g$, then $-g$ is larger in absolute value than $a\Phi(g)$ and $\Phi(g)$ as well, so the same argument as in Section 3 implies that coercion reduces social welfare.

Finally, we note that we have not analyzed the case in which both $\Phi(g) > g$ for some g and (IIR) is binding. In such cases, Corollary 9 does not apply and thus does not rule out the possibility that coercion may sometimes be good for social welfare. Nevertheless, the combination of Corollary 9, which implies that when $\Phi(g) \leq g$, coercion reduces social welfare, and Corollary 5, which implies that coercion also reduces social welfare when $\Phi(g)$ is sufficiently large to make (IIR) slack, suggests

that any example in which coercion increases social welfare will be somewhat special. This is the basis of our claim that, under fairly weak assumptions, our general welfare result that coercion reduces utilitarian social welfare appears robust to the presence of interim participation constraints.

5.5 Substitutability Between Effort and Productivity

All of our results so far have been predicated on the complementarity between productivity and effort. This is natural: greater effort increases the probability of success, and success is more valuable to producers with greater probability. Nevertheless, one can imagine more general setups where higher productivity producers may have less need for effort. In this subsection, we briefly discuss how our analysis can be adapted to this case, even though we find it less natural and suspect that it is also less relevant empirically. To allow for this possibility, suppose that, instead of ax, the expected output of a producer of type x when the agent exerts effort a is given by a twice continuously differentiable function $y(a,x)$. Suppose also that the producer can only condition payments and punishments on a binary signal, which equals *High* with probability a and equals *Low* otherwise.[45] With this formulation, given (a,g,w^h,w^l), the principal's expected payoff is

$$Py(a,x) - aw^h - (1-a)\,w^l - \eta\chi(g).$$

It can now be verified that if $\partial^2 y(a,x)/\partial a\partial x > 0$ for all a and x, then productivity and effort continue to be complements, the problem of characterizing equilibrium contracts is supermodular in (a,g,x), and all of our results generalize. Suppose, instead, that $\partial^2 y(a,x)/\partial a\partial x < 0$ for all a and x, so that productivity and effort are substitutes. Equilibrium contracts are still characterized by the maximization problem

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} Py(a,x) - a\left[(1-a)\,c'(a) + c(a) + \bar{u} - g\right]_+ - (1-a)\left[-ac'(a) + c(a) + \bar{u} - g\right]_+ - \eta\chi(g), \tag{23}$$

which is now supermodular in $(a,g,-x,P,-\bar{u},-\eta)$. Thus, the only difference between (7) and (23), for the purposes of comparative statics, is that x is complementary with $(a,g,P,-\bar{u},-\eta)$ in (7), but is substitutable with $(a,g,P,-\bar{u},-\eta)$ in (23). This discussion thus establishes the following (proof omitted):

Proposition 17 *Suppose that $\partial^2 y(a,x)/\partial a\partial x < 0$ for all a and x. Then the set of equilibrium contracts for a producer of type x forms a lattice, and the greatest and smallest equilibrium contracts $(a^+(x), g^+(x))$ and $(a^-(x), g^-(x))$ are decreasing in x. The remaining comparative statics in Proposition 2 apply.*

[45] Implicitly, this formulation assumes that the principal cannot condition payments and punishments on the actual realization of output. While this is not very appealing, it is the simplest way to introduce the possibility that effort and productivity may be substitutes, which is our main focus in this subsection.

In this case workers are *better* off when matched with more productive producers, as these producers demand less effort and thus use less coercion. One implication is that, in contrast to the analysis in subsection 4.6, slave trade, conditional on coercive labor relations, now increases agents' welfare rather than reducing it. Having noted this possibility, we still emphasize that, to us, the case with complements appears both more natural and empirically more plausible, though distinguishing between the two cases is ultimately an empirical matter.

6 Concluding Remarks

Standard economic models assume that transactions in the labor market are "free". For most of human history, however, the bulk of labor transactions have been "coercive," in the sense that the threat of force was essential in convincing workers to take part in the employment relationship, and in accepting the terms of employment. The small existing literature on slavery and forced labor does not model what we view as the fundamental effect of coercion on labor transactions—coercion makes workers accept employment relations (terms of employment) that they would otherwise reject. This paper provides a tractable model incorporating this feature and uses it to provide a range of comparative static results useful for understanding when coercive labor relations are more likely to arise and what their welfare implications are.

At the heart of our model is the principal-agent relationship between a potentially coercive producer and an agent. Coercion, modeled as investment by the producer in "guns," affects the participation constraint of the agent. We first analyzed this principle-agent problem and derived partial equilibrium comparative statics, and then embedded it in a general equilibrium framework to study the relationship between labor scarcity/abundance and labor coercion. Both the partial and general equilibrium analyses are made tractable because of the supermodularity of the maximization problem characterizing equilibrium contracts. This supermodularity is not only mathematically convenient, but also economically central: coercion covaries positively with the amount of effort that the principal wishes to extract from the agent. Intuitively, greater effort implies that the principal will have to reward the agent more frequently because success is more likely. But this also implies that greater coercion, which reduces these rewards, becomes more valuable to the principal. An implication of this observation is that agents with better outside options will be coerced less. Another is that agents with higher marginal productivities of labor in the coercive sector will be coerced more.

We show that, consistent with the famous work by Fogel and Engerman (1974), coercion increases effort. However, our formulation also implies that, in contrast to Fogel and Engerman's interpretation, this does not imply that slavery (or coercion) is or may be "efficient". On the contrary, coercion always reduces (utilitarian) social welfare, because the structure of the principal-agent model dictates that coercion hurts the agent more than the additional effort it leads to helps the principal. Our model also shows that coercion changes both workers' and producers' incentives to make ex ante investments in their relationships, and points out a new channel via which trading coerced workers makes them

worse off.

A major question in the economics of coercion, both from a historical perspective and for understanding the continued prevalence of forced labor in many developing societies today, is the effect of labor scarcity/abundance on coercion. In a seminal article on the Caribbean plantation complex, Domar (1970) argued that labor scarcity encouraged coercion (by increasing the wage, which the employers could save by using slavery). In contrast, the most famous theories of the decline of feudalism and serfdom link these events to the sharp reductions in population experienced in Western Europe starting in the second half of the 15th century. Acemoglu, Johnson and Robinson (2002), in turn, suggested that Europeans were more likely to set up coercive institutions in colonies with abundant labor, because setting up such institutions was not profitable when labor was scarce. Our general equilibrium analysis shows why these diverse perspectives are not contradictory. Labor scarcity first creates a labor demand effect: it increases the marginal product of workers in the coercive sector, and thus encourages employers to use greater coercion and extract higher effort from their workers. It also creates an outside option effect: it increases the outside option of the workers in the noncoercive sector, and reduces coercion because employers demand lower effort and use less coercion with workers that have greater outside options. Domar's argument for Caribbean slavery applies when the labor demand effect is strong and the outside option effect is limited. In contrast, when the outside option effect is important, perhaps as in Western Europe in the 16th century because of the presence of well-established cities, labor scarcity will discourage coercion. Finally, a slight modification of our model, where coercion investments are made before matching, also leads to the result that labor scarcity may discourage coercion through an economies of scale effect (because coercing workers becomes more costly when labor is scarce).

We view this paper as a first step towards a systematic analysis of coercion in the labor market and its implications for the organization of production and economic development. Despite the historical and current importance of forced labor and other coercive relations, many central questions in the economics of coercion have not previously been answered or even posed. Our framework provides a first set of answers to these questions and can serve as a starting point for different directions of study. Theoretical and empirical investigations of the dynamics of coercion, of why coercive relationships persist in many developing countries even today, of the effects of coercion on technology choices and organizational decisions, and of how coercive production impacts trade are important areas for future research (see, for example, Naidu and Yuchtman, 2009, for a recent empirical study of labor coercion). A particularly fruitful area of future research is a more in-depth analysis of the politics of coercion. We presumed the presence of an institutional environment that permitted coercion by producers. In many instances, coercion comes to an end, or is significantly curtailed, when political forces induce a change in the institutional environment. Combining our microeconomic model of coercion with a model of endogenous institutions would be one way of making progress in this direction.

Appendix A: Proofs of Propositions 1 and 2, and Lemma 3

Proof of Proposition 1

Suppose first that an equilibrium contract involves $a > 0$. Recall that with two possible outcomes, the first-order approach is valid (see Appendix B for more on this), so the incentive compatibility constraint can be written in the form of (IC_2), which also exploits the fact that, given our assumption $\lim_{a\to 1} c'(a) = \infty$, $a < 1$, and thus the first-order condition must hold as equality. Then the producer's problem can be written as

$$\max_{(a,g,w^h,w^l,p^h,p^l)\in[0,1]\times\mathbb{R}^5_+} a\left(Px - w^h\right) + (1-a)\left(-w^l\right) - \eta\chi(g)$$

subject to

$$a\left(w^h - p^h\right) + (1-a)\left(w^l - p^l\right) - c(a) \geq \bar{u} - g \tag{IR$_0$}$$

and

$$w^h - p^h - \left(w^l - p^l\right) = c'(a). \tag{IC$_2$}$$

It is suboptimal to set both $w^h > 0$ and $p^h > 0$, or $w^l > 0$ and $p^l > 0$, as reducing both w^h (w^l) and p^h (p^l) by ε would strictly increase profits without affecting the constraints, so $w^h = \left[u^h\right]_+$ and $w^l = \left[u^l\right]_+$. The problem becomes

$$\max_{(a,g,u^h,u^l)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}^2} a\left(Px - \left[u^h\right]_+\right) - (1-a)\left[u^l\right]_+ - \eta\chi(g) \tag{A-1}$$

subject to

$$au^h + (1-a)\,u^l - c(a) \geq \bar{u} - g \tag{IR$_1$}$$

and

$$u^h - u^l = c'(a). \tag{IC$_1$}$$

Substituting (IC_1) into (IR_1) yields

$$u^h - (1-a)\,c'(a) - c(a) \geq \bar{u} - g. \tag{IR$_2$}$$

(IR_2) must bind, as otherwise decreasing u^h would increase profit. Using (IC_1) and (IR_2) to substitute u^h and u^l out of (A-1) now yields (7).

It remains only to be shown that $g > 0$, $u^l \leq 0$, and $u^h \geq 0$, and finally that $a > 0$ in any equilibrium contract (that $w^h = (1-a)\,c'(a) + c(a) + \bar{u} - g > 0$ and $p^l = -ac'(a) + c(a) + \bar{u} - g \geq 0$ then following immediately from (IR_2) and (IC_1)). First, the result that $g > 0$ in any equilibrium contract with $a > 0$ follows from (7), since it must be that $\chi'(g) \in \left[\frac{a}{\eta}, \frac{1}{\eta}\right]$ in any equilibrium contract. The result that $u^l \leq 0$ and $u^h \geq 0$ is established in the next lemma.

Lemma 6 *In any equilibrium contract with $a > 0$, we have $u^l \leq 0$ and $u^h \geq 0$.*

Proof. Note first that the Lagrangian for (A-1) subject to (IR_2) and (IC_1) is

$$a\left(Px-\left[u^h\right]_+\right)-(1-a)\left[u^l\right]_+-\eta\chi(g)+\lambda\left(au^h+(1-a)\,u^l-c(a)-(\bar{u}-g)\right)+\mu\left(u^h-u^l-c'(a)\right). \tag{A-2}$$

Now suppose to obtain a contradiction that $u^l>0$. Then $\left[u^l\right]_+=u^l$ and is differentiable. Moreover, in this case $u^h>0$, as $u^h>u^l$ (since $a>0$), and thus $\left[u^h\right]_+=u^h$ is also differentiable. Moreover, clearly, $\frac{d\left[u^h\right]_+}{du^h}=\frac{d\left[u^l\right]_+}{du^l}=1$. Then differentiating (A-2) with respect to u^h and u^l, we obtain

$$1=\lambda+\frac{\mu}{a} \tag{FOCu^h}$$

and

$$1=\lambda-\frac{\mu}{1-a}. \tag{FOCu^l}$$

These first-order conditions always hold, as setting u^l or u^h to ∞ or $-\infty$ cannot be optimal if $a\in(0,1)$. Combining (FOCu^h) and (FOCu^l) then implies that $\mu=0$.

Now differentiating (A-2) with respect to a and using (IC_1) yields

$$Px-\left(\left[u^h\right]_+-\left[u^l\right]_+\right)=\mu c''(a). \tag{FOCa}$$

(FOCa) holds with equality by our assumptions that $a>0$ and that $\lim_{a\to1}c'(a)=\infty$. The fact that $\mu=0$ then implies that $\left[u^h\right]_+-\left[u^l\right]_+=u^h-u^l=Px$. Substituting $\left[u^h\right]_+-\left[u^l\right]_+=u^h-u^l=Px$ and $\mu=0$ into the Lagrangian (A-2), we obtain

$$-u^l-\eta\chi(g)+\lambda\left(u^l+Px-c(a)-(\bar{u}-g)\right).$$

This expression is maximized at the corner $a=0$, contradicting our assumption that $a>0$. This completes the proof of claim that $u^l\leq0$.

Finally, to show $u^h\geq0$, suppose to obtain a contradiction that $u^h<0$. Then (A-2) is differentiable with respect to u^h at the optimum, and its derivative with respect to u^h is

$$a\lambda+\mu.$$

Since $a>0$, this can equal 0 only if $\lambda=0$ and $\mu=0$. But then the maximum of (A-2) over $a\in[0,1]$ is attained at $a=1$, which implies that the producer's profit equals $-\infty$, by (IR_2) and our assumption that $\lim_{a\to1}c'(a)=\infty$, which yields a contradiction. ■

It remains only to check that a solution to the producer's problem with $a>0$ exists if $Px>\bar{u}+c'(0)$. Consider the producer's problem as first choosing a and then maximizing profit given a. The producer's optimal profit given fixed a is continuous in a. Therefore, if we show that the producer's optimal profit given a is increasing in a for all sufficiently small $a>0$, we will have shown that $a=0$ is not a solution.

The producer's problem, given $a > 0$, is maximizing (7) over $g \in \mathbb{R}_+$:

$$\max_{g \in \mathbb{R}_+} a \left(Px - \left[(1-a)\,c'(a) + c(a) + \bar{u} - g\right]_+\right) - (1-a)\left[-ac'(a) + c(a) + \bar{u} - g\right]_+ - \eta\chi(g). \quad \text{(A-3)}$$

The right-derivative of (A-3) with respect to a is

$$Px - \left(\left[u^h\right]_+ - \left[u^l\right]_+\right) - a(1-a)\,c''(a)\left(\frac{d\left[u^h\right]_+}{du^h} - \frac{d\left[u^l\right]_+}{du^l}\right), \quad \text{(A-4)}$$

where the $d\left[u^h\right]_+/du^h$ and $d\left[u^l\right]_+/du^l$ denote right-derivatives. We have shown above that $u^h \geq 0$ and $u^h > u^l$ at an optimum with $a > 0$, and $d\left[u^h\right]_+/du^h - d\left[u^l\right]_+/du^l \leq 1$, so (A-4) is no less than $Px - u^h - a(1-a)\,c''(a)$. By (IR$_2$), as a converges to 0, u^h converges to at most $\bar{u} + c'(0)$. Therefore, provided that $Px > \bar{u} + c'(0)$, (A-4), and thus the derivative of (A-3) with respect to a, is strictly positive for sufficiently small a. This establishes that $a > 0$, and completes the proof of the Proposition. ■

Proof of Proposition 2

The maximization problem (7) is weakly supermodular in $(a, g, x, P, -\bar{u}, -\eta)$ everywhere. Therefore, the results that the set of equilibrium contracts for a producer of type x forms a lattice, and that these greatest and smallest equilibrium contracts, $(a^+(x), g^+(x))$ and $(a^-(x), g^-(x))$, are nondecreasing in x and P and nonincreasing in $\bar{u}$ and η follow immediately (e.g., using Theorems 2.7.1 and 2.8.1 in Topkis (1998)).

In the rest of the proof, we show that these comparative static results are strict in the sense that whenever we have a change from $(x, P, \bar{u}, \eta)$ to $(x', P', \bar{u}', \eta')$, where $x' \geq x$, $P' \geq P$, $\bar{u}' \leq \bar{u}$, and $\eta' \leq \eta$, with at least one strict inequality, we have "strictly increasing" instead of "nondecreasing" and "strictly decreasing" instead of "nonincreasing". In the process of doing this, we will also establish the "genericity" result claimed in Remark 3. These results are stated in the following lemma, the proof of which completes the proof of the proposition.

Lemma 7 *Consider the maximization problem (7), let $a(x, P, \bar{u}, \eta)$ and $g(x, P, \bar{u}, \eta)$ denote the least [or greatest] solution to (7), then $a(x', P', \bar{u}', \eta') > a(x, P, \bar{u}, \eta)$ and $g(x', P', \bar{u}', \eta') > g(x, P, \bar{u}, \eta)$ for any $(x', P', \bar{u}', \eta')$ where $x' \geq x$, $P' \geq P$, $\bar{u}' \leq \bar{u}$, and $\eta' \leq \eta$, with at least one strict inequality. Moreover, let $u^l(x, P, \bar{u}, \eta)$ be a solution at the parameter vector $(x, P, \bar{u}, \eta)$. If $u^l(x, P, \bar{u}, \eta) \geq 0$, then $u^l(x', P', \bar{u}', \eta') < 0$.*

Proof. For brevity, we will prove this lemma for a change in x, holding P, $\bar{u}$, and η constant. The argument for the other cases is analogous.

Consider a change from x to $x' > x > \underline{x}$ (where recall that $\underline{x}$ is the lower support of the distribution F). The first part of Proposition 2, which has already been established, implies that $a(x', P', \bar{u}', \eta') \geq a(x, P, \bar{u}, \eta)$ and $g(x', P', \bar{u}', \eta') \geq g(x, P, \bar{u}, \eta)$, which we shorten to $a(x') \geq a(x)$ and $g(x') \geq g(x)$.

It is then straightforward to verify that if $u^l(x) < 0$, then (7) can be written as

$$(a, g) \in \arg \max_{(\tilde{a},\tilde{g})\in[0,1]\times\mathbb{R}_+} Px\tilde{a} - \tilde{a}\left[(1-\tilde{a})\,c'(\tilde{a}) + c(\tilde{a}) + \bar{u} - \tilde{g}\right]_+ - \eta\chi(\tilde{g}),$$

which is strictly supermodular in x (or more generally in $(x, P, \bar{u}, \eta)$) in the neighborhood of x, and moreover g is given by (10). Then since $a > 0$, and thus $g > 0$, a and g cannot remain constant following an increase in x, and since they cannot decline, it must be the case that an increase in x strictly increases a and g.

Therefore, any change from x to $x' > x$ will give weak comparative comparative static results only if $u^l(\tilde{x}) = 0$ for all $\tilde{x} \in [x, x']$ (from Lemma 6, we have $u^l(\tilde{x}) \leq 0$, so $u^l(\tilde{x}) > 0$ is not possible). We will next establish that $u^l(x) = 0$ (and thus weak (but not strict) supermodularity is only possible for infinitesimal changes at $x = \underline{x}$). Suppose, to obtain a contradiction, that this is the case, and in particular, that for the least solutions in both cases, we have $a(x) = a(x')$ (the exact same argument applies to greatest solutions). Then since $x > \underline{x}$, we must have $u^l(\tilde{x}) = 0$ in the neighborhood of x. This implies, from (IC_1) and (IR_2), that

$$u^l = -ac'(a) + c(a) + \bar{u} - g = 0. \tag{A-5}$$

Consider variations in a and g along (A-5) (i.e., holding $u^l = 0$). Since $a > 0$ and c is differentiable, this implies

$$\frac{dg}{da} = -ac''(a) < 0.$$

Using this expression, we obtain that if $a(x) = a(x')$, then $g(x) = g(x')$. Next, since $a(x)$ and $g(x)$ are optimal, any such variation along (A-5) should not increase the value of (7). This is only possible if

$$Px - a(x)\,c''(a(x)) - (1-a(x))\,c'(a(x)) - c(a(x)) - \bar{u} + g(x) + \eta\chi'(g(x))\,a(x)\,c''(a(x)) = 0, \tag{A-6}$$

where we have used the fact that $u^h > 0$ wherever $u^l = 0$, by (IC_1) and the fact that $a > 0$. Now repeating the same argument for x', and recalling that by hypothesis we have $u^l(x') = 0$ and $a(x) = a(x')$, and thus $g(x) = g(x')$ (as just established), we also have

$$Px' - a(x)\,c''(a(x)) - (1-a(x))\,c'(a(x)) - c(a(x)) - \bar{u} + g(x) + \eta\chi'(g(x))\,a(x)\,c''(a(x)) = 0. \tag{A-7}$$

However, in view of the fact that $x' > x$, (A-7) cannot be true at the same time as (A-6). This implies that (i) either $u^l(x) < 0$ or $u^l(x') < 0$, or (ii) $a(x') > a(x)$ ($a(x) > a(x')$ is ruled out by weak supermodularity). But if $a(x') > a(x)$ and $u^l(x') = u^l(x)$, then (A-5) implies that $g(x') < g(x)$, contradicting the fact that $g(x') \geq g(x)$. So in either case we have $u^l(x') < 0$ if $u^l(x) = 0$. Since this is true for any $x' > x$, we can never have $u^l(x) = 0$ and $u^l(x') = 0$ for any $x' > x$, and thus we can only have $u^l(x) = 0$ at $x = \underline{x}$. Moreover, as $u^l(x') < 0$, strict comparative statics apply at $x = x'$. Since this is true for any pair of x, $x' > x$, we conclude that (i) all comparative statics are strict (even if $u^l(x) = 0$, we must have $u^l(x') < 0$, and thus a change from x to x' will strictly increase a and g), and (ii) for any pair of $x' > x$, if $u^l(x) = 0$, we must have $u^l(x') < 0$, and thus $u^l = 0$ is only possible at one level of x, that is, at the lowest level $x = \underline{x}$. ■

Proof of Lemma 3

Fix $t' \geq t$. That $F(x,t)$ and $F(x,t')$ have fixed points follows from Kakutani's Fixed Point Theorem. Since $F(x,t)$ and $F(x,t')$ have closed graphs, as does the 45° line, the intersections of $F(x,t)$ and $F(x,t')$ with 45° lines are closed, so $F(x,t)$ and $F(x,t')$ have smallest and greatest fixed points.

Let y be the smallest fixed point of $F(x,t)$. For any $x < y$, every element of $F(x,t)$ is strictly greater than x, as otherwise applying Kakutani's Fixed Point Theorem to the interval $[0,x]$ would imply that $F(x,t)$ has a fixed point smaller than x, and thus smaller than y. By the definition of the weak set order, this implies that every element of $F(x,t')$ is strictly greater than x, for any $x < y$. So the smallest fixed point of $F(x,t')$ is no smaller than y.

Let z be the greatest fixed point of $F(x,t)$. Then either $z \in F(z,t')$ or every element of $F(z,t')$ is strictly greater than z. In the latter case, applying Kakutani's Fixed Point Theorem to the interval $[z,\bar{x}]$ implies that $F(x,t')$ has a fixed point greater than z. In either case, the greatest fixed point of $F(x,t')$ is no smaller than z. ■

Appendix B: Multiple Output Levels

In this Appendix, we allow for an arbitrary finite number of outputs. Let $f(y|a)$ is the probability that output equals y given effort a, and assume that $f(y|a)$ is twice-differentiable with respect to a. Let $\bar{y}$ be the highest possible output and let $\underline{y}$ be the lowest possible output. We normalize the price of output to 1 and do not consider producer heterogeneity.[46] Equilibrium contracts are given by the solution to the following problem:

$$\max_{(w(\cdot),p(\cdot),a,g)} \sum_y (y - w(y)) f(y|a) - \eta\chi(g),$$

subject to

$$\sum_y (w(y) - p(y)) f(y|a) - c(a) \geq \bar{u} - g, \tag{IR$_{B0}$}$$

and

$$a \in \arg\max_{a\in[0,1]} \sum_y (w(y) - p(y)) f(y|a) - c(a). \tag{IC$_{B0}$}$$

We continue to focus on the case where equilibrium contracts involve $a > 0$.[47] Then, from Theorem 1 in Jewitt (1988), the first-order approach to this problem is valid provided that:

1. $\sum_{z\leq\bar{z}} \sum_{y\leq z} f(y|a)$ is nonincreasing and convex in a for each $\bar{z}$;

[46] It is straightforward to see that if producers differ according to productivity parameter x and a producer with productivity x produces output xy with probability $f(y|a)$, all supermodularity results from Section 3 will continue to hold.

[47] We do not spell out the assumptions on primatives under which equilibrium contracts involve $a > 0$. Assumption 1 suffices for this in the two-outcome case, and similar sufficient conditions can be developed for the case with multiple output levels, but this is orthogonal to our focus here.

2. $\sum_{y\leq\bar{z}} yf(y|a)$ is nondecreasing and concave in a for each $\bar{z}$;

3. $\frac{f_a(y|a)}{f(y|a)}$ is nondecreasing and concave in y for every a.

Jewitt this provides an interpretation of these conditions. Note Condition 3 implies the usual (and relatively weak) monotone likelihood ratio property (MLRP). Therefore, MLRP holds throughout this appendix. Jewitt argues that the remainder of the third condition is the most restrictive; in addition to MLRP, "[it requires that] variations in output at higher levels are relatively less useful in providing "information" on the agents effort than they are at lower levels of output," (Jewitt, 1988, p. 1181). Note that Jewitt's condition on utility functions is not needed when the agent is risk-neutral.

Given these three conditions, and simplifying the exposition by imposing $u^l < 0$,[48] we can apply the first-order approach and (writing $u(y)$ for $w(y) - p(y)$) rewrite the problem as

$$\max_{(w(\cdot),p(\cdot),a,g)} \sum_y \left(y - [u(y)]_+\right) f(y|a) - \eta\chi(g)$$

subject to

$$\sum_y u(y)\, f(y|a) - c(a) \geq \bar{u} - g \tag{IR$_{B1}$}$$

and

$$\sum_y u(y)\, f_a(y|a) = c'(a). \tag{IC$_{B1}$}$$

The associated Lagrangian is

$$\sum_y \left[\left(y - [u(y)]_+\right) f(y|a) - \eta\chi(g) + \lambda\left(u(y)\, f(y|a) - c(a) - (\bar{u} - g)\right) + \mu\left(u(y)\, f_a(y|a) - c'(a)\right)\right].$$

Differentiating under the integral with respect to $u(y)$ and rearranging implies that

$$1 = \lambda + \mu\frac{f_a(y|a)}{f(y|a)} \tag{B-1}$$

if $u(y) > 0$,

$$0 = \lambda + \mu\frac{f_a(y|a)}{f(y|a)} \tag{B-2}$$

if $u(y) < 0$, and

$$0 \leq \lambda + \mu\frac{f_a(y|a)}{f(y|a)} \leq 1 \tag{B-3}$$

if $u(y) = 0$. By MLRP, (B-1), (B-2), and (B-3) imply that $u(y) = 0$ for all $y \notin \{\underline{y}, \bar{y}\}$, $u(\underline{y}) \leq 0$, and $u(\bar{y}) \geq 0$. This is a standard "bang-bang" result given MLRP and the agent's risk-neutrality. To simplify notation, let $u^h \equiv u(\bar{y})$ and let $u^l \equiv u(\underline{y})$, paralleling the notation in subsection 3.1.

[48] Under suitable restrictions on $f(y|a)$, which we neglect here for ease of exposition, $u_l < 0$ in any equilibrium contract for "generic" parameter values, as in the two-outcome case considered in the text.

The producer's maximization problem can now be written as

$$\max_{(a,g,u^h,u^l)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}^2} \sum_y yf(y|a) - f(\bar{y}|a)\left[u^h\right]_+ - \eta\chi(g)$$

subject to

$$f(\bar{y}|a)u^h + f(\underline{y}|a)u^l - c(a) \geq \bar{u} - g \tag{IR$_{B2}$}$$

and

$$f_a(\bar{y}|a)u^h + f_a(\underline{y}|a)u^l = c'(a). \tag{IC$_{B2}$}$$

(IC$_{B2}$) can be rewritten as

$$u^l = \frac{c'(a) - f_a(\bar{y}|a)u^h}{f_a(\underline{y}|a)}. \tag{B-4}$$

Substituting (B-4) into (IR$_{B2}$) and using the fact that (IR$_{B2}$) binds at the solution gives

$$\left(f(\bar{y}|a) - \frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)}f(\underline{y}|a)\right)u^h + \frac{f(\underline{y}|a)}{f_a(\underline{y}|a)}c'(a) - c(a) = \bar{u} - g,$$

which may be rewritten as

$$u^h = \frac{c(a) - \frac{f(\underline{y}|a)}{f_a(\underline{y}|a)}c'(a) + \bar{u} - g}{f(\bar{y}|a) - \frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)}f(\underline{y}|a)}. \tag{B-5}$$

Finally, substituting (B-5) into the principal's objective gives

$$\max_{(a,g)\in[0,1]\times\mathbb{R}_+} \sum_y yf(y|a) - f(\bar{y}|a)\left[\frac{c(a) - \frac{f(\underline{y}|a)}{f_a(\underline{y}|a)}c'(a) + \bar{u} - g}{f(\bar{y}|a) - \frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)}f(\underline{y}|a)}\right]_+ - \eta\chi(g), \tag{B-6}$$

To establish the supermodularity of (B-6) in $(a, g, -\bar{u}, -\eta)$, it suffices to show that the cross-partial of the right-hand side of (B-6) with respect to a and g is always nonnegative. This is immediate if $\left[u^h\right]_+ = 0$. If $\left[u^h\right]_+ > 0$, the right-hand side of (B-6) may be rewritten as

$$\sum_y yf(y|a) - \left(\frac{1}{1 - \left(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)}\right)/\left(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)}\right)}\right)\left(c(a) - \frac{f(\underline{y}|a)}{f_a(\underline{y}|a)}c'(a) + \bar{u} - g\right) - \eta\chi(g), \tag{B-7}$$

where we have divided the numerator and denominator of the middle term by $f_a(\bar{y}|a)$. The cross-partial of (B-7) with respect to a and g is nonnegative if and only if the derivative of $\left(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)}\right)/\left(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)}\right)$ with respect to a is nonnegative. We have thus established the following result:

Proposition 18 *Suppose that $a > 0$, Conditions 1—3 hold, and $\left(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)}\right)/\left(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)}\right)$ is increasing in a. Then equilibrium contracts are given by (B-6), and (B-6) is supermodular in $(a, g, -\bar{u}, -\eta)$.*

The condition that $\left(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)}\right) / \left(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)}\right)$ is increasing in a is neither particularly restrictive nor particularly easy to interpret. To see why it is sufficient for supermodularity of equilibrium contracts, note that, by (B-5) (which is determined by (IR_{B_2}) and (IC_{B_2})), increasing g by Δ allows the principal to reduce w^h (i.e., payment after the highest output level) by $\frac{\Delta}{f(\bar{y}|a)-\frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)}f(\underline{y}|a)}$. Since the principal pays w^h with probability $f(\bar{y}|a)$, increasing g by Δ benefits the principal by $\frac{f(\bar{y}|a)}{f(\bar{y}|a)-\frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)}f(\underline{y}|a)}$, which is increasing in a if $\left(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)}\right) / \left(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)}\right)$ is. It is instructive to compare this with the following slight generalization of the two-outcome case discussed in Section 3.1: Suppose there are only two outcomes $\underline{y} < \bar{y}$, but that $f(\bar{y}|a)$ need not equal a. Then, $f(\bar{y}|a) - \frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)}f(\underline{y}|a) = f(\bar{y}|a) + f(\underline{y}|a) = 1$ (as $f_a(\bar{y}|a) = -f_a(\underline{y}|a)$), so $\frac{\Delta}{f(\bar{y}|a)-\frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)}f(\underline{y}|a)} = \Delta$. Therefore, increasing g by Δ benefits the principal by $f(\bar{y}|a)$, which is increasing in a under MLRP. In particular, we see that $\left(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)}\right) / \left(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)}\right)$ is increasing in a if $f(\bar{y}|a) + f(\underline{y}|a)$ does not depend on a, regardless of the number of outcomes.

Appendix C: Existence of Equilibrium in the General Case

In this appendix, we use the distributional approach to equilibrium in games with a continuum of players to establish existence of "mixed-strategy equilibria" in the general model of Section 4, and show that analogs of Propositions 6 and 7 apply to extremal mixed-strategy equilibria. The analysis also shows that our approach in the text carries over to the case where $F(\cdot)$ has mass points. We establish existence of a mixed-strategy equilibrium using Theorem 1 of Mas-Collel (1984), but applying this result requires us to recast the maximization problem characterizing equilibrium contracts in an equivalent form.

In our original formulation of the producer's problem, we viewed a producer with productivity x as choosing $(a, g) \in \mathbb{R}^2_+$ to maximize $aPx - a\left[(1-a)c'(a) + c(a) + \bar{u} - g\right]_+ - \eta\chi(g)$, and we also assumed that the distribution of productivity F had a continuous density f. Both of these assumptions were adopted to simplify the notation in the text and can be relaxed. This appendix outlines how this can be done.

First, note that since $\lim_{g\to\infty}\chi(g) = \infty$ and $a \leq 1$, there exists a positive number $\bar{g}$ such that setting $g > \bar{g}$ is dominated for any producer. We now specify that a producer with productivity x chooses $(q, g) \in [0, \bar{x}] \times [0, \bar{g}]$ to maximize

$$\pi(x) \equiv qP - \frac{q}{x}\left[\left(1-\frac{q}{x}\right)c'\left(\frac{q}{x}\right) + c\left(\frac{q}{x}\right) + \bar{u} - g\right]_+ - \left(1-\frac{q}{x}\right)\left[-\frac{q}{x}c'\left(\frac{q}{x}\right) + c\left(\frac{q}{x}\right) + \bar{u} - g\right]_+ - \eta\chi(g), \tag{C-1}$$

where we have rewritten a as $\frac{q}{x}$. By our assumption that $\lim_{a\to 1} c'(a) = \infty$, any solution to the problem of a producer with productivity x satisfies $q < x$. It is also clear that (C-1) is continuous in (p, q).

The point of the above reformulation is to give each producer the same action set, and our assumption that $\lim_{a\to 1} c'(a) = \infty$ ensures that every equilibrium contract in the reformulated game is feasible in the original game. Given this, existence of a mixed-strategy equilibrium (or "Cournot-Nash equilibrium distribution" in the language of Mas-Collel) is immediate from Theorem 1 of Mas-Collel. Formally, a *mixed-strategy equilibrium* is now a measure τ over $\pi(x) \times ([0,\bar{x}] \times [0,\bar{g}])$ such that the marginals τ_π and $\tau_{(q,g)}$of τ on $\pi(x)$ and $[0,\bar{x}] \times [0,\bar{g}]$, respectively, satisfy

1. $\tau_\pi = f$,
2. $\tau(\{\pi(x), q, g\} : q \leq x) = 1$, and
3. $\tau\left(\{\pi(x), q, g\} : \pi\left(x, q, g, \tau_{(q,g)}\right) \geq \pi\left(x, \tilde{q}, \tilde{g}, \tau_{(q,g)}\right) \text{ for all } (\tilde{q}, \tilde{g}) \in [0,\bar{x}] \times [0,\bar{g}]\right) = 1,$

where $\pi\left(x, q, g, \tau_{(q,g)}\right)$ is payoff to a producer with productivity x who chooses (q,g) where other producers' actions are given by the distribution $\tau_{(q,g)}$. We also write $\pi(x, q, g, Q, G)$ for the payoff to a producer with productivity x who chooses (q,g) when facing aggregates Q and G, as the distribution $\tau_{(q,g)}$ affects each producer's payoff only through Q and G.

Proposition 19 *A mixed-strategy equilibrium exists.*

Proof. The proof follows closely the proof of Mas-Collel's Theorem 1, with the addition that each distribution consistent with producers' playing best responses satisfies point 2 above (since $q < x$ in any equilibrium contract of a producer with productivity x, by our assumption that $\lim_{a\to 1} c'(a) = \infty$), so any fixed point must satisfy point 2 as well. ■

Finally, we show that analogs of Propositions 6 and 7 apply to mixed-strategy equilibria of the kind considered above. The argument is identical to that leading to Propositions 6 and 7, adapted to the distributional definition of equilibria—we therefore omit some details. Given a joint density $\tau_{(q,g)}$ over $[0,\bar{x}] \times [0,\bar{g}]$, let τ_q and τ_g be the corresponding marginal densities over $[0,\bar{x}]$ and $[0,\bar{g}]$, respectively. Let

$$Q\left(\tau_{(q,g)}\right) \equiv \int_0^{\bar{x}} q\tau_q(q)\, dq,$$

and let

$$G\left(\tau_{(q,g)}\right) \equiv \int_0^{\bar{g}} g\tau_g(g)\, dg,$$

the corresponding levels of the aggregates. Let

$$\hat{\phi}'(Q, \gamma, L, \tilde{u}, \eta) = \left\{ \begin{array}{c} Q' : \text{there exists a density } \tau \text{ on } \pi(x) \times ([0,\bar{x}] \times [0,\bar{g}]) \text{ such that} \\ \tau\left(\{\pi(x), q, g\} : \pi\left(x, q, g, Q, G\left(\tau_{(q,g)}\right)\right) \geq \pi\left(x, \tilde{q}, \tilde{g}, Q, G\left(\tau_{(q,g)}\right)\right)\right) = 1 \\ \text{and } Q' = Q\left(\tau_{(q,g)}\right) \end{array} \right\}.$$

That is, $\hat{\phi}'$ maps Q and parameter values to those Q' that are mixed-strategy equilibrium levels of output in the modified game where price is fixed at $P(LQ)$. It is clear that the set of fixed points of $\hat{\phi}'$ equals the set of mixed-strategy equilibrium values of Q. We may now establish the following:

Lemma 8 *The greatest and smallest elements of* $\hat{\phi}'(Q, \gamma, L, \tilde{u}, \eta)$ *are increasing in* $F(x)$ *and* γ*, and decreasing in* Q*,* L*,* $\tilde{u}$*, and* η*.*

Proof. Analogous to proof of Lemma 4. ■

Proposition 20 *The greatest and smallest mixed-strategy equilibrium values of* Q *are increasing in* $F(x)$ *and* γ*, and decreasing in* L*,* $\tilde{u}$*, and* η*.*

Proof. Analogous to proof of Proposition 6. ■

Repeating the above argument interchanging the roles of Q and G gives the following:

Proposition 21 *The greatest and smallest mixed-strategy equilibrium values of* G *are increasing in* $F(x)$ *and* γ*, and decreasing in* L*,* $\tilde{u}$*, and* η*.*

Propositions 20 and 21 show that the results of Section 4.3 continue to apply when we allow for mixed strategies, which addresses the issue of equilibrium existence.

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